Financial Risk Forecasting
Chapter 6
Analytical value-at-risk for options and bonds

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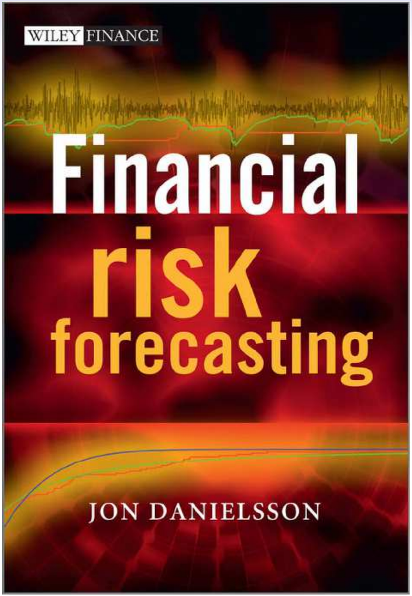
To accompany
Financial Risk Forecasting
www.financialriskforecasting.com
Published by Wiley 2011

Version 3.0, August 2017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The focus of this chapter

- Calculate VaR for options and bonds
    - Not possible with methods from Chapters 4 and 5
- We start by using analytical methods, deriving VaR mathematically
- Monte Carlo methods are discussed in Chapter 7
    - Preferred for most applications

VaR for options and bonds

- Chapters 4 and 5 showed how a VaR can be obtained an asset distribution
- That is not possible for assets such as bonds and options, as their intrinsic value changes with passing of time
    - e.g. the price of bond converges to fixed value as time to maturity elapses, so inherent risk decreases over time
- Value of bonds and options is non–linearly related to the underlying asset

Organization

- The first two sections of these slides introduce the problem of the nonlinear relationship between the underlying asset and a bond and option
- The last two sections show how one can use mathematical approximations to obtain a closed form solution
- Generally, such methods are not recommended
- And is better to use the simulation methods in the next chapter

Notation

T	Delivery time/maturity
r	Annual interest rate
σ_r	Volatility of daily interest rate increments
σ_a	Annual volatility of an underlying asset
σ_d	Daily volatility of an underlying asset
τ	Cash flow
D^*	Modified duration
C	Convexity
Δ	Option delta
Γ	Option gamma
$g(\cdot)$	Generic function name for pricing equation
ϑ	Portfolio value

Bonds

Bond pricing

- A bond is a fixed income instrument
- Typially with regular payments
- Bond price is given by *present value* of *future cash flows*

$$\sum_{t=1}^{T} \frac{\tau_t}{(1+r_t)^t}$$

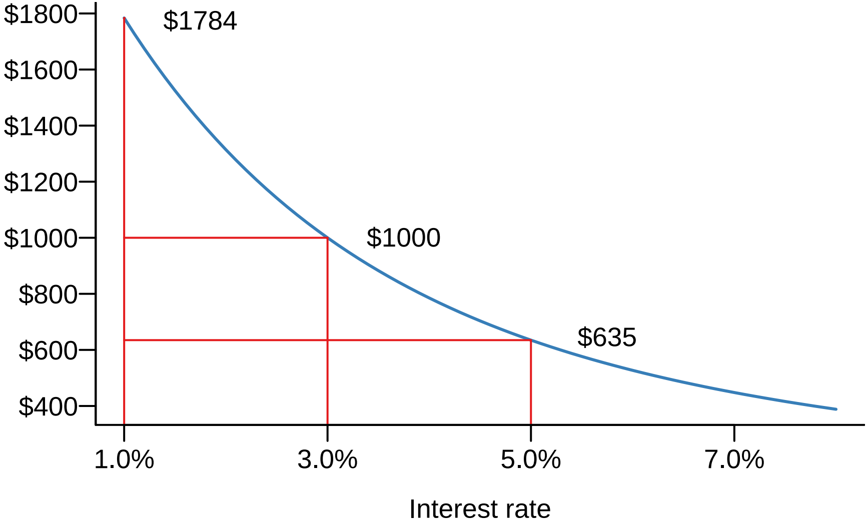
- Where $\{\tau_t\}_{t=1}^{T}$ includes the coupon and principal payments
- And r_t is the interest rate in each period

Bond risk asymmetry

- Bond has face value \$1000, maturity of 50 years and annual coupon of \$30
- Yield curve is flat, annual interest rates at 3%
- So its current price is equal to the par value
- Now consider parallel shifts in the yield curve to 1% or 5%

Interest rate	Price	Change in price
1%	\$1784	\$784
3%	\$1000	
5%	\$635	-\$365

Bond risk asymmetry



Bond risk

- Change from 3% to 1% makes bond price increase by \$784
- Change from 3% to 5% makes it fall by \$365

Options

Options

- An option gives its owner the right, but not the obligation, to *call* (buy) or *put* (sell) an underlying asset at a *strike price* on a fixed expiry date
  - European options can only be exercised at expiration
  - American options can be exercised at any point up to expiration
- We will focus on European options, but the basic analysis could be extended to many other variants

Black-Scholes equation

Pricing European options

- Black and Scholes (1973) developed an equation for pricing European options
- Refer to the Black-Scholes (BS) pricing function as $g(\cdot)$
- We use the following notation:

P_t	Price of underlying asset at year t
X	Strike price
r	Annual risk-free interest rate
$T-t$	Time until expiration
σ_a	Annual volatility
Φ	Standard normal distribution

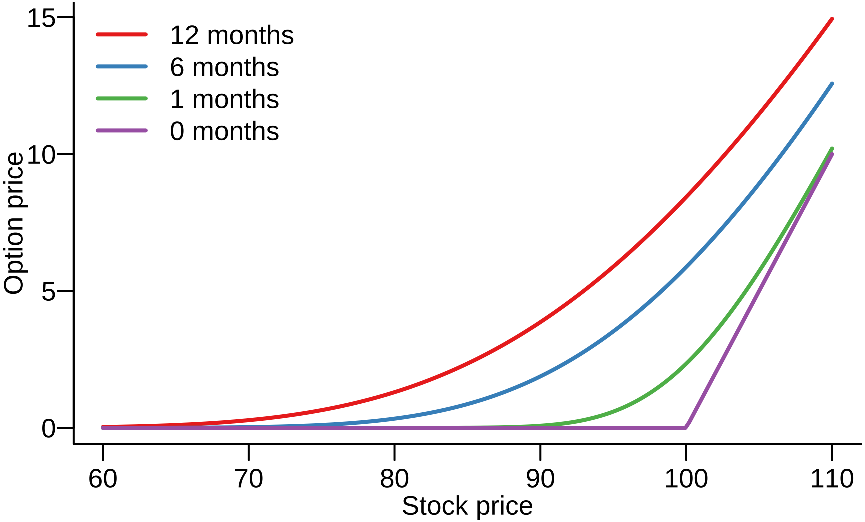
- The BS function for an European option

$$\text{put}_t = Xe^{-r(T-t)} - P_t + \text{call}_t$$
$$\text{call}_t = P_t\Phi\left(d_1\right) - Xe^{-r(T-t)}\Phi\left(d_2\right)$$

where

$$d_1 = \frac{\log\left(P_t/X\right) + \left(r + \sigma_a^2/2\right)\left(T-t\right)}{\sigma_a\sqrt{T-t}}$$
$$d_2 = \frac{\log\left(P_t/X\right) + \left(r - \sigma_a^2/2\right)\left(T-t\right)}{\sigma_a\sqrt{T-t}}$$
$$= d_1 - \sigma_a\sqrt{T-t}$$

- Value of an option is affected by many underlying factors
- Standard BS assumptions:
  - Flat nonrandom yield curve
  - The underlying asset has continuous IID-normal returns
- Our objective is to map risk in the underlying asset onto an option
  - This can be done using the option *Delta* and *Gamma*


12 months
6 months
1 months
0 months
15
10
5
0
Option price
60
70
80
90
100
110
Stock price

VaR for bonds

- There are several ways to approximate bond risk as a function of risk in interest rates
- One way is to use Ito's lemma, another to follow the derivation for options
- Here we only present the result, as a formal derivation would just repeat the one given for options

Modified duration

- We define *modified duration*, D^*, as the negative first derivative of the bond-pricing function, $g'(r)$, divided by prices:

$$D^* = -\frac{1}{P} g'(r)$$

- Modified duration measures price sensitivity of a bond to interest rate movements

Duration-normal VaR

Two steps to calculate bond VaR

1. Identify the distribution of interest rate changes, dr
2. Map distribution onto bond prices

Duration-normal VaR

- We assume the distribution of interest rate changes is given by

$$r_t - r_{t-1} = dr \sim \mathcal{N}\left(0, \sigma_r^2\right)$$

  but we could use almost any distribution
- Regardless of whether we use Ito's lemma or follow the derivation for options, we arrive at the duration-normal method to get bond VaR
- Here we find that bond returns are simply modfied duration times interest rate changes so

$$R_{\text{Bond}} \overset{\text{Approximately}}{\sim} \mathcal{N}\left(0, \left(\sigma_r D^*\right)^2\right)$$

Duration-normal VaR

- Now the VaR follows directly:

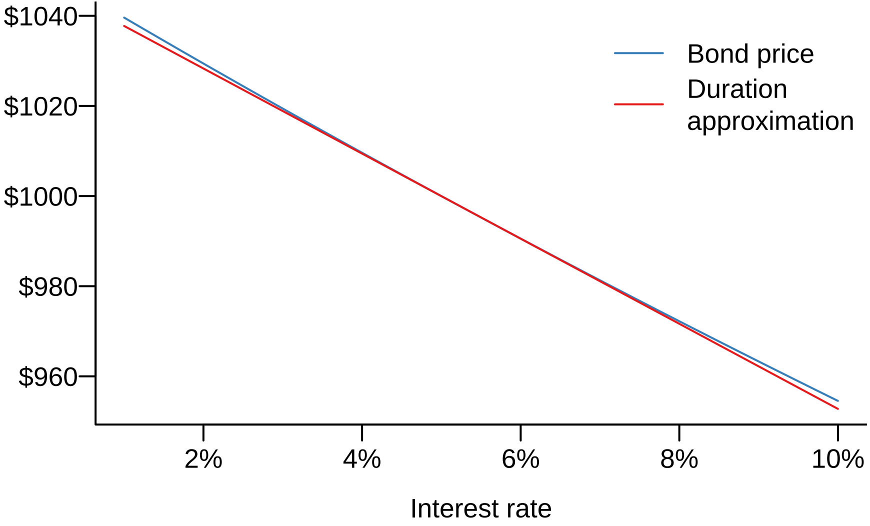
$$\text{VaR}_{\text{Bond}}(p) \approx D^* \times \sigma_r \times \Phi^{-1}(p) \times \vartheta$$

Accuracy of duration-normal VaR

- The accuracy of these approximations depends on magnitude of duration and the VaR time horizon
- Main sources of error are assumptions of linearity and flat yield curve
- We now explore these issues graphically

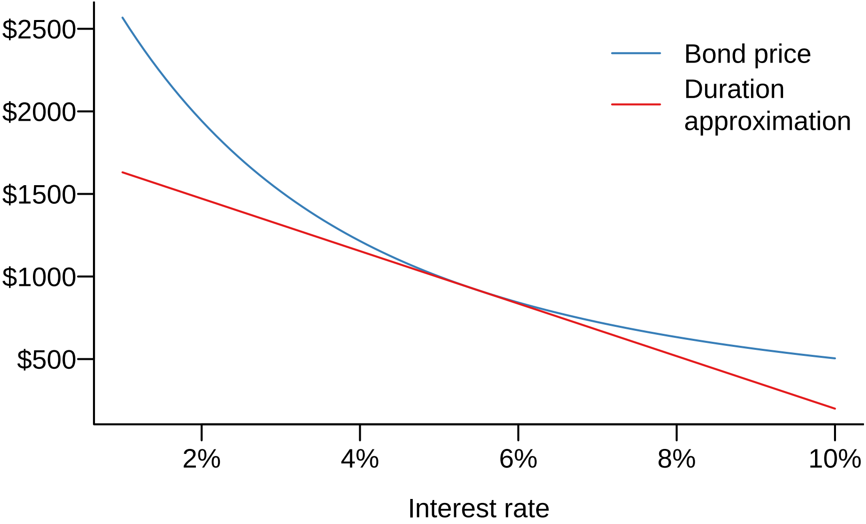
Bond prices and duration

Accuracy of duration approximation for T=1



Bond prices and duration

Accuracy of duration approximation for T=50



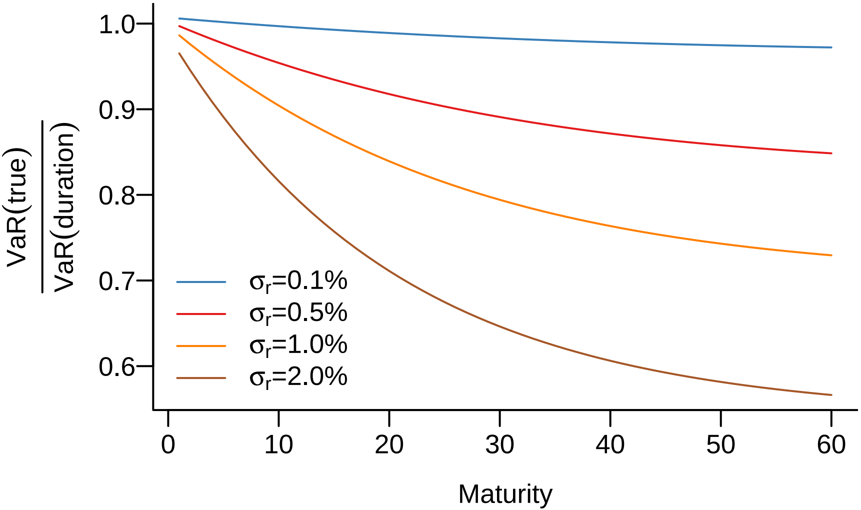
Bond prices and duration

Accuracy of duration approximation for T=1 and T=50

- The graphs compare bond prices and duration approximation for two maturities, $T = 1$ and $T = 50$
- It is clear that duration approximation is quite accurate for short-dated bonds, but very poor for long-dated ones
- We conclude that maturity is a key factor when it comes to accuracy of VaR calculations using duration-normal methods

Error in duration-normal VaR

Various volatilities of interest rate changes



Error in duration-normal VaR

Higher volatility of interest rate changes leads to larger error

- The graph on the previous slide shows how the accuracy of duration-normal VaR is affected by interest rate change volatility
- Duration-normal VaR is compared with VaR (true), which is calculated with a Monte Carlo simulation
- Looking at maturities from 1 year to 60 years and volatility from 0.1% to 2.0%, we see that the error in duration-normal VaR increases as volatility of interest rate changes increases

Accuracy of duration-normal VaR

- Based on these observations, we conclude that duration-normal VaR approximation is best for short-dated bonds and low volatilities
- Quality declines sharply with increased volatility and longer maturities

Convexity and VaR

- Straightforward to improve duration approximation by adding second-order term, thereby allowing for convexity
- However, even after incorporating convexity there is often considerable bias in VaR calculations
- Adding higher order terms increases mathematical complexity, especially if we have a portfolio of bonds
- For these reasons, Monte Carlo methods are generally preferred

Delta

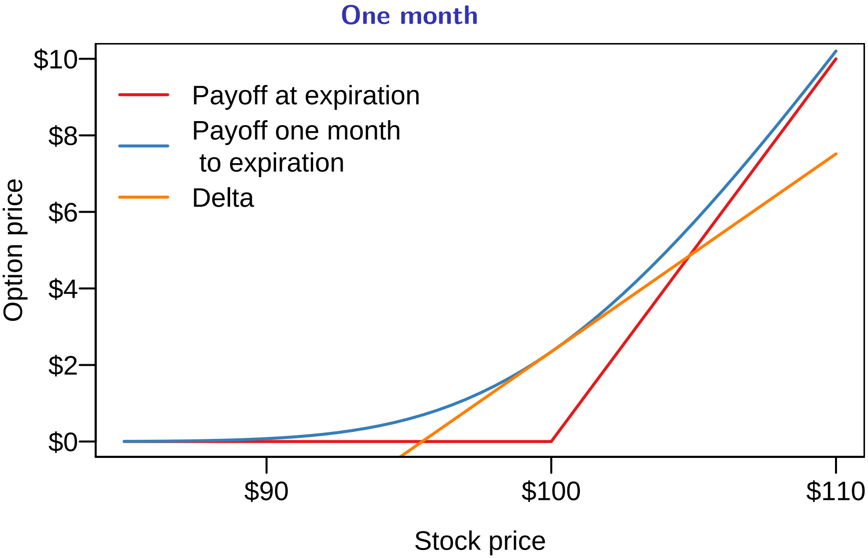
- First-order sensitivity of an option with respect to the underlying price is called delta, defined as:

$$\Delta = \frac{\partial g(P)}{\partial P} = \begin{cases} \Phi\left(d_1\right) > 0 & \text{call} \\ \Phi\left(d_1\right) - 1 < 0 & \text{put} \end{cases}$$

- Delta is equal to ± 1 for deep-in-the-money options (depending on whether it is call or put), close to ± 0.5 for at-the-money options and 0 for deep out-of-the-money options

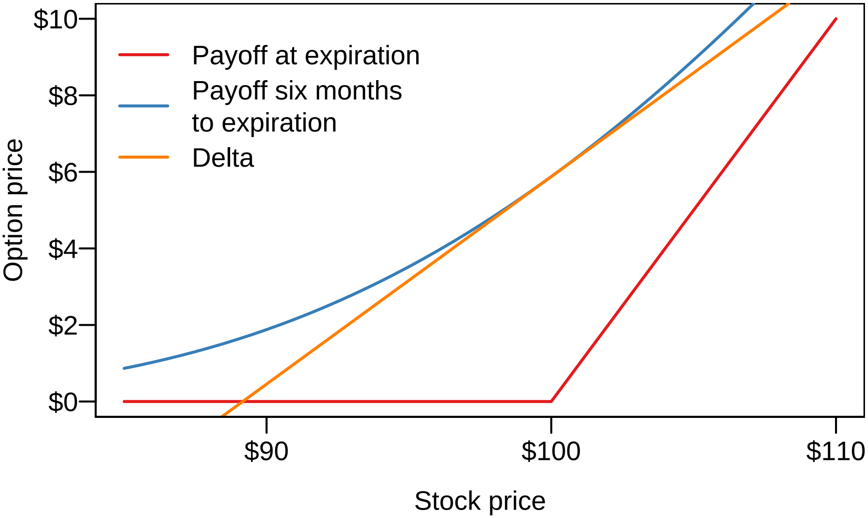
- A small change in P changes the option price by approximately Δ, but the approximation gets gradually worse as the deviation of P becomes larger
- We can graph the price of a call option for a range of strike prices and two different maturities to gauge the accuracy of the delta approximation
- We let $X = 100$, $r = 0.01$ and $\sigma = 0.2$ and compare maturities of one and six months

Accuracy of Delta approximation



Accuracy of Delta approximation

Six months



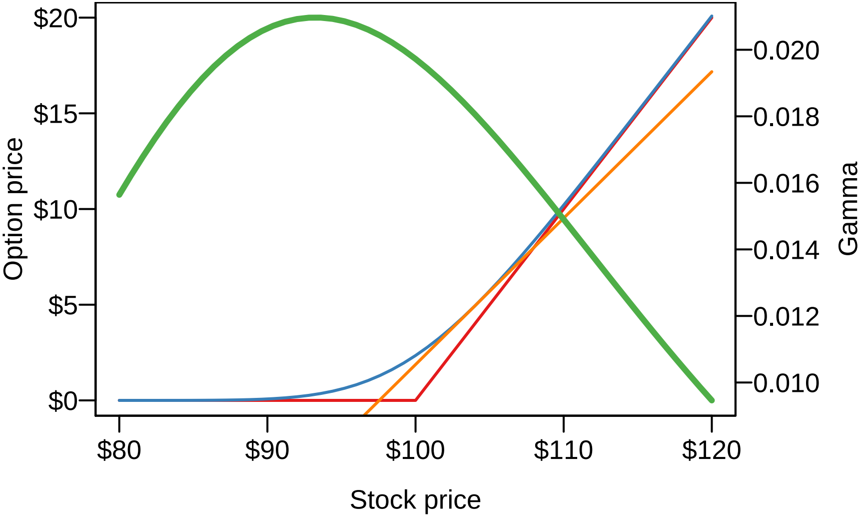
Gamma

- Second-order sensitivity of an option with respect to the underlying price is called gamma, defined as:

$$\Gamma = \frac{\partial^2 g(P)}{\partial P^2} = e^{-r(T-t)} \frac{\Phi\left(d_1\right)}{P_t \sigma_a \sqrt{(T-t)}}$$

- Gamma is highest when an option is a little out of the money and dropping as the underlying price moves away from the strike price
- We can see this by adding a plot of gamma to the previous graph of option price with one month to expiry
  - Not surprising since the price plot increasingly becomes a straight line for deep in-the-money and out-of-the-money options

Gamma for the one month option



Numerical example

- Consider an option that expires in six months ($T = 0.5$) with strike price $X = 90$, price $P = 100$ and 20% volatility
- Let $r = 5\%$ be the risk-free rate of return
- The call delta is 0.8395 and the put delta is -0.1605
- The gamma is 0.01724

Delta-normal VaR

- We can use delta to approximate changes in the option price as a function of changes in the price of the underlying
- Denote daily change in stock prices as:

$$dP = P_t - P_{t-1}$$

- The price change dP implies that the option price will change approximately by

$$dg = g_t - g_{t-1} \approx \Delta dP = \Delta \left(P_t - P_{t-1} \right)$$

where Δ is the option delta at time $t-1$; and g is either the price of a call or put

- Simple returns on the underlying are

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

and following the BS assumptions, they are IID-normal with daily volatility σ_d:

$$R_t \sim \mathcal{N}\left(0, \sigma_d^2\right)$$

- The derivation of VaR for options parallels the one for simple returns in Chapter 5

Delta-normal VaR

Derivation of VaR for options

- Denote $\text{Var}_o(p)$ as the VaR of an option, where p is probability:

$$
\begin{aligned}
p &= \Pr\left(g_t - g_{t-1} \leq -\text{VaR}_o(p)\right) \\
&= \Pr\left(\Delta\left(P_t - P_{t-1}\right) \leq -\text{VaR}_o(p)\right) \\
&= \Pr\left(\Delta P_{t-1} R_t \leq -\text{VaR}_o(p)\right) \\
&= \Pr\left(\frac{R_t}{\sigma_d} \leq -\frac{1}{\Delta}\frac{\text{VaR}_o(p)}{P_{t-1}\sigma_d}\right)
\end{aligned}
$$

Delta-normal VaR

Derivation of VaR for options

- Now it follows that the VaR for holding an option on one unit of the asset is:

$$\text{VaR}_o(p) \approx -|\Delta| \times \sigma_d \times \Phi_R^{-1}(p) \times P_{t-1}$$

- This means that the option VaR is simply δ multiplied by the VaR of the underlying, VaR_u:

$$\text{VaR}_o(p) \approx |\Delta| \, \text{VaR}_u(p)$$

- We need absolute value because we may have put or call options and VaR is always positive

Quality of Delta-normal VaR

- The quality of this approximation depends on the extent of nonlinearities
  - Better for shorter VaR horizons
- For risk management purposes, poor approximation of delta to the true option price for large changes in the price of the underlying is clearly a cause of concern

Delta and Gamma

- We can also approximate the option price by the second-order expansion, Γ
- Since dP is normal, $(dP)^2$ is chi-squared
- The same issues apply here as for bonds: Adding higher orders increases complexity a lot, without eliminating bias

Summary

- We have seen that forecasting VaR for options and bonds is much more complicated than for basic assets like stocks and foreign exchange
- The mathematical complexity in this chapter is not high, but the approximations have low accuracy
- To obtain higher accuracy the mathematics become much more complicated, especially for portfolios
- This is why the Monte Carlo approaches in Chapter 7 are preferred in most practical applications