Financial Risk Forecasting
Chapter 9
Extreme Value Theory

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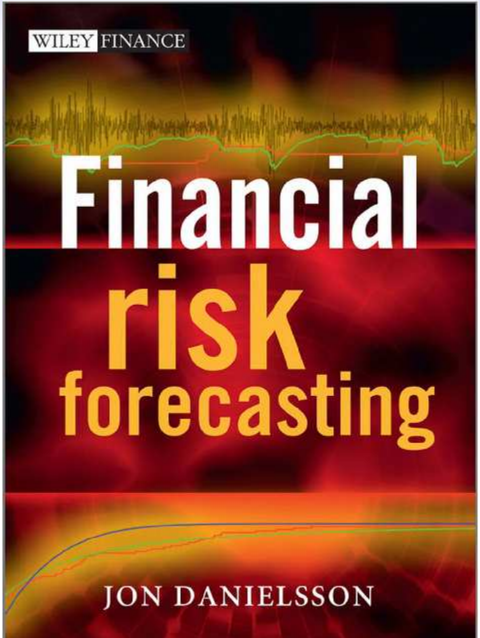
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The Focus of This Chapter

- Basic introduction to extreme value theory (EVT)
- Asset returns and fat tails
- Applying EVT
- Aggregation and convolution
- Time dependence

Notation

ι	Tail index
$\xi = 1/\iota$	Shape parameter
M_T	Maximum of X
C_T	Number of observations in the tail
u	Threshold value
ψ	Extremal index

Extreme Value Theory

Types of Tails

- In this book, we follow the convention of EVT being presented in terms of the *upper tails* (ie *positive observations*)
- In most risk analysis we are concerned with the *negative observations* in the lower tails, hence to follow the convention, we can *pre-multiply returns by -1*
- Note, the upper and lower tails do not need to have the same thickness or shape

Extreme Value Distributions

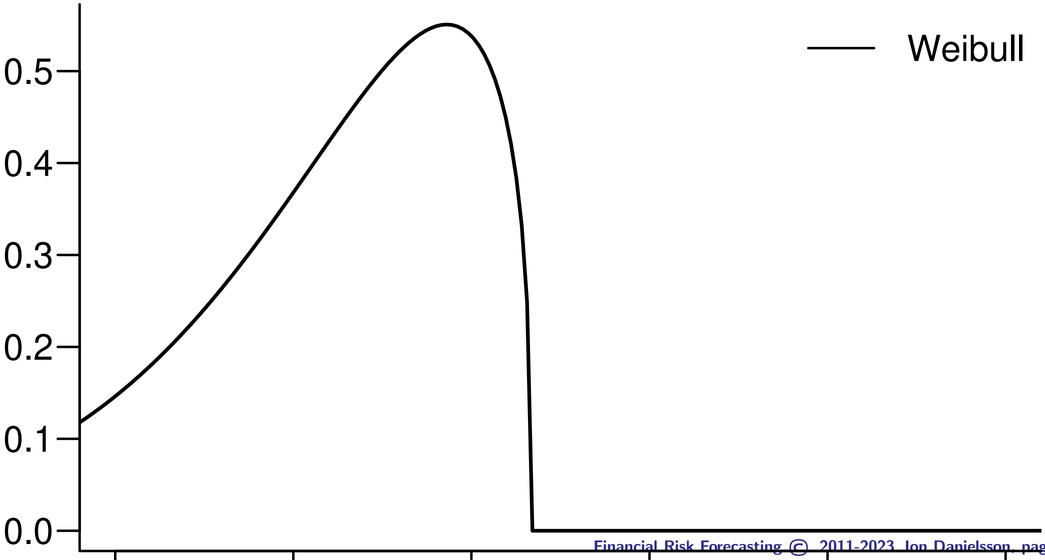
- In most risk applications, we do not need to focus on the entire distribution
- The main result of EVT states that the tails of all distributions fall into one of three categories, regardless of the overall shape of the distribution
  - See next slide for the three distributions
- Note, this is true given the distribution of an asset return does not change over time

Weibull Thin tails where the distribution has a finite endpoint (eg the distribution of mortality and insurance/re-insurance claims)

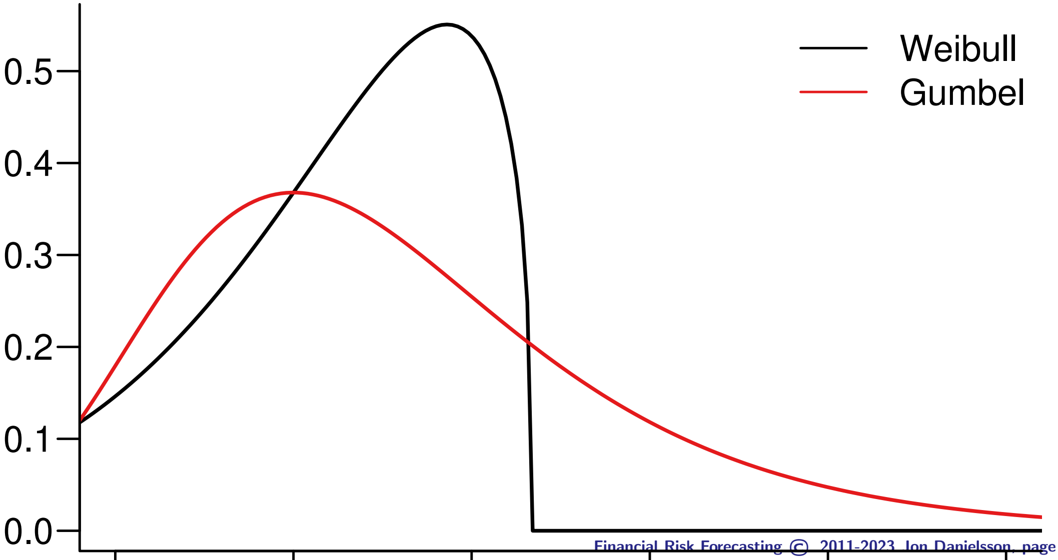
Gumbel Tails decline exponentially (eg the normal and log-normal distributions)

Fréchet Tails decline by a *power law*; such tails are know as "fat tails" (eg the Student-t and Pareto distributions)

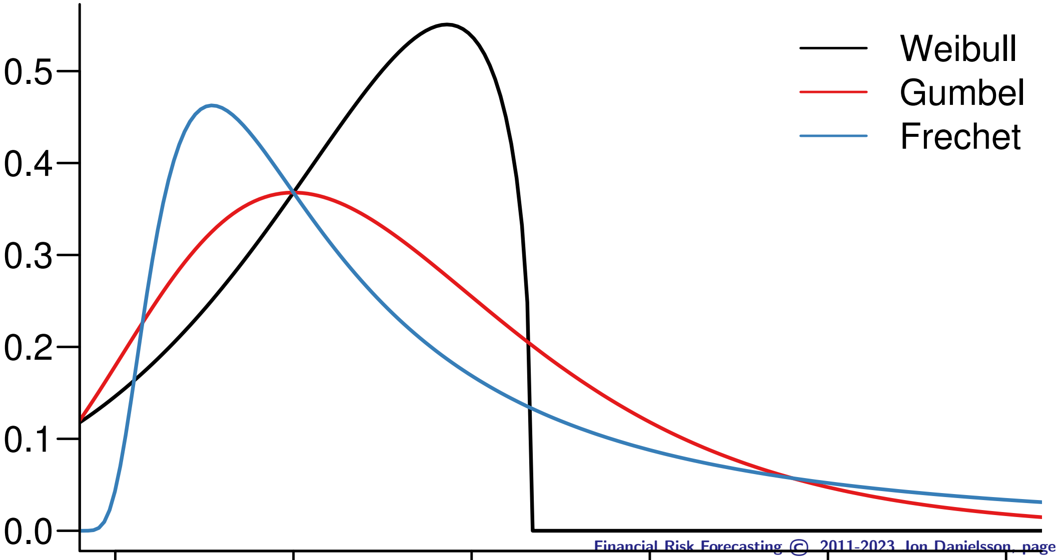
Extreme Value Distributions



Extreme Value Distributions



Extreme Value Distributions



Fréchet distribution

- From the last slide, the Weibull clearly has a finite endpoint
- And the Fréchet tail is thicker than the Gumbel's
- In most applications in finance, we know that returns are fat tailed
- Hence we limit our attention to the Fréchet case

Generalised Extreme Value Distribution

- The Fisher and Tippett (1928) and Gnedenko (1943) theorems are the fundamental results in EVT
- The theorems state that the maximum of a sample of properly normalised IID random variables *converges in distribution* to one of the three possible distributions: the Weibull, Gumbel or the Fréchet

Generalised Extreme Value Distribution

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- An alternative way of stating this is in terms of the maximum domain of attraction(MDA)
- MDA is the set of limiting distributions for the properly normalised maxima as the sample size goes to infinity

Fisher-Tippet and Gnedenko Theorems

- Let $X_1, X_2, ..., X_T$ denote IID random variables (RVs) and the term M_T indicate maxima in sample of size T
- The *standardised distribution* of maxima, M_T, is

$$\lim_{T \to \infty} \Pr \left\{ \frac{M_T - a_T}{b_T} \leq x \right\} = H(x)$$

where the constants a_T and $b_T > 0$ exist and are defined as $a_T = T\mathbb{E}(X_1)$ and $b_T = \sqrt{\mathrm{Var}(X_1)}$

Fisher-Tippet and Gnedenko Theorems

- Then the limiting distribution, $H(.)$, of the maxima as the *generalised extreme value (GEV)* distribution is

$$H_\xi(x) = \begin{cases} \exp\left\{-(1+\xi x)^{-\frac{1}{\xi}}\right\}, & \xi \neq 0 \\ \exp\left\{-\exp(-x)\right\}, & \xi = 0 \end{cases}$$

Limiting Distribution $H_\xi(.)$

- Depending on the value of ξ, $H_\xi(.)$ becomes one of the three distributions:
  - if $\xi > 0$, $H_\xi(.)$ is the **Fréchet**
  - if $\xi < 0$, $H_\xi(.)$ is the **Weibull**
  - if $\xi = 0$, $H_\xi(.)$ is the **Gumbel**

Asset Returns and Fat Tails

Fat Tails

- The term *"fat tails"* can have several meanings, the most common being *"extreme outcomes occur more frequently than predicted by normal distribution"*
- While such a statement might make intuitive sense, it has little mathematical rigor as stated
- The most frequent definition one may encounter is Kurtosis, but it is not always accurate at indicating the presence of fat tails ($\kappa > 3$)
- This is because kurtosis is more concerned with the sides of the distribution rather than the *heaviness of tails*

A Formal Definition of Fat Tails

- The formal definition of fat tails comes from *regular variation*

Regular variation A random variable, X, with distribution $F(.)$ has fat tails if it varies regularly at infinity; that is there exists a positive constant ι such that:

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\iota}, \quad \forall x > 0, \iota > 0$$

Tail Distributions

- In the fat-tailed case, the tail distribution is Fréchet:

$$H(x) = \exp(-x^{-\iota})$$

Lemma *A random variable X has regular variation at infinity (ie has fat tails) if and only if its distribution function F satisfies the following condition:*

$$1 - F(x) = \mathbb{P}\{X > x\} = Ax^{-\iota} + o(x^{-\iota})$$

for positive constant A, when $x \to \infty$

Tail Distributions

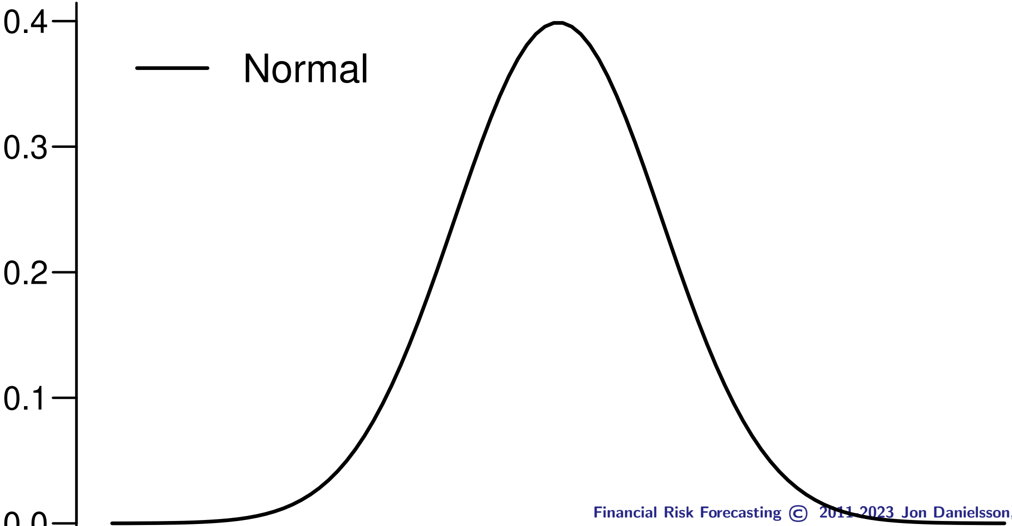
- The expression $o(x^{-\iota})$ is the *remainder term* of the Taylor-expansion of $\Pr\{X > x\}$, it consists of terms of the type Cx^{-j} for constant C and $j > \iota$
- As $x \to \infty$, the tails are asymptotically Pareto- distributed:

$$F(x) \approx 1 - Ax^{-\iota}$$

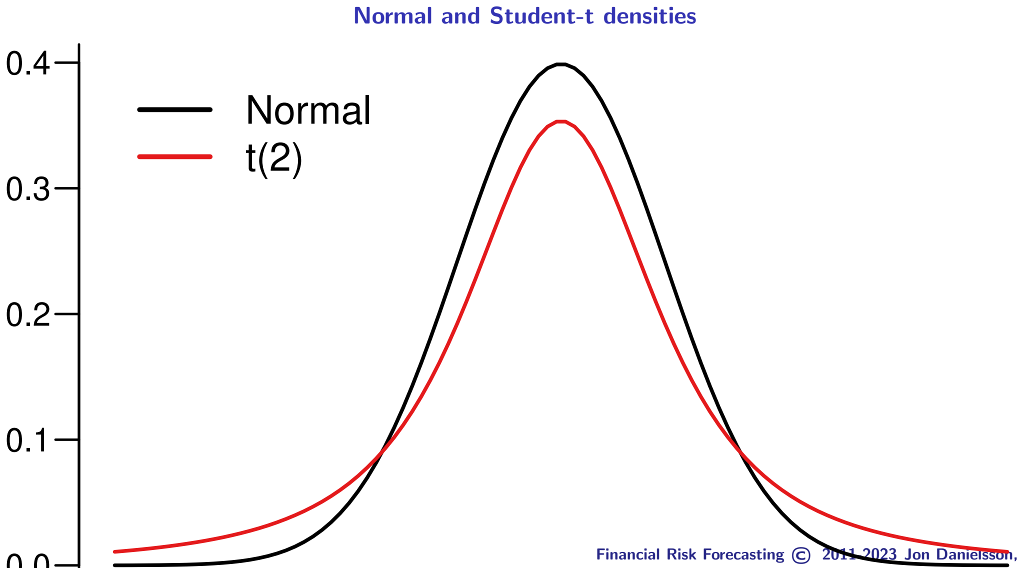
where $A > 0$; $\iota > 0$; and $\forall x > A^{1/\iota}$

Normal and Fat Distributions

Normal and Student-t densities

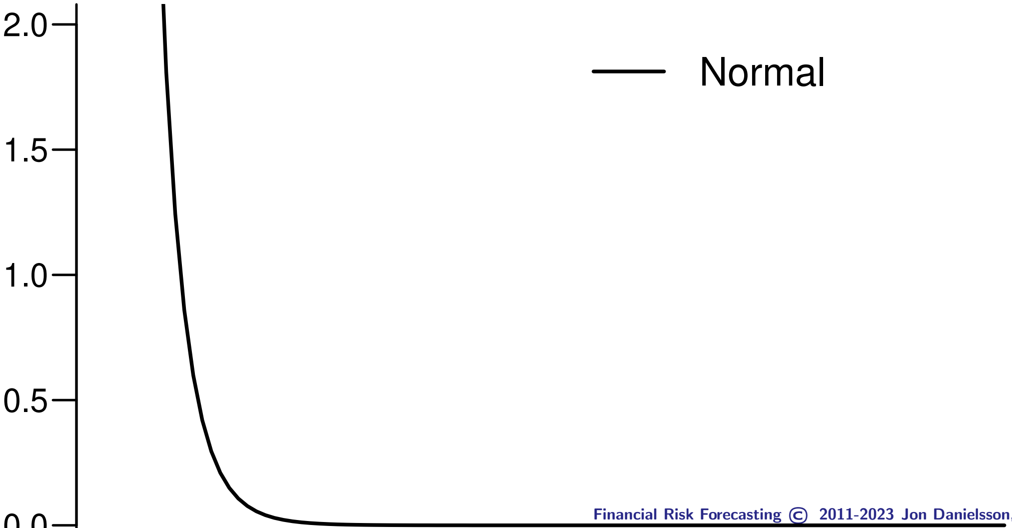


Normal and Fat Distributions



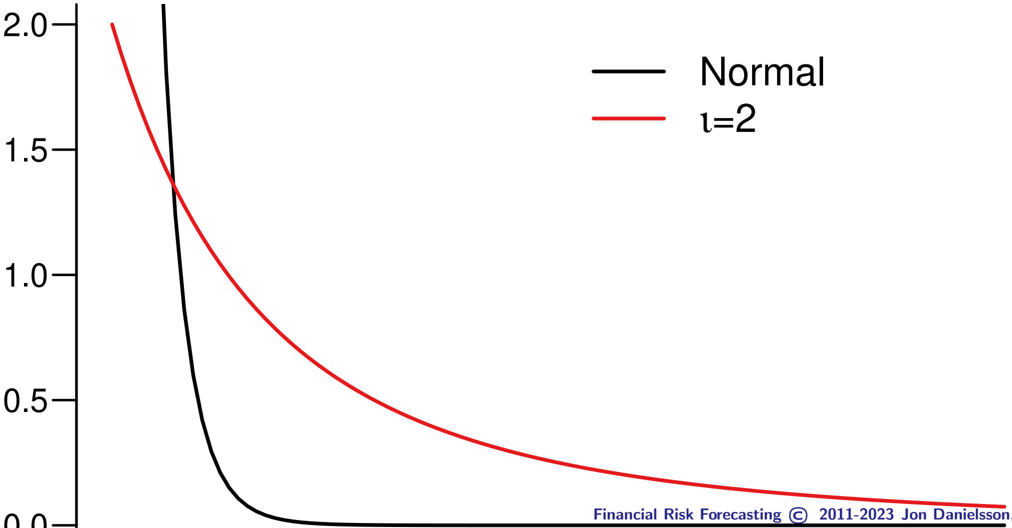
Normal and Fat Distributions

Pareto tails



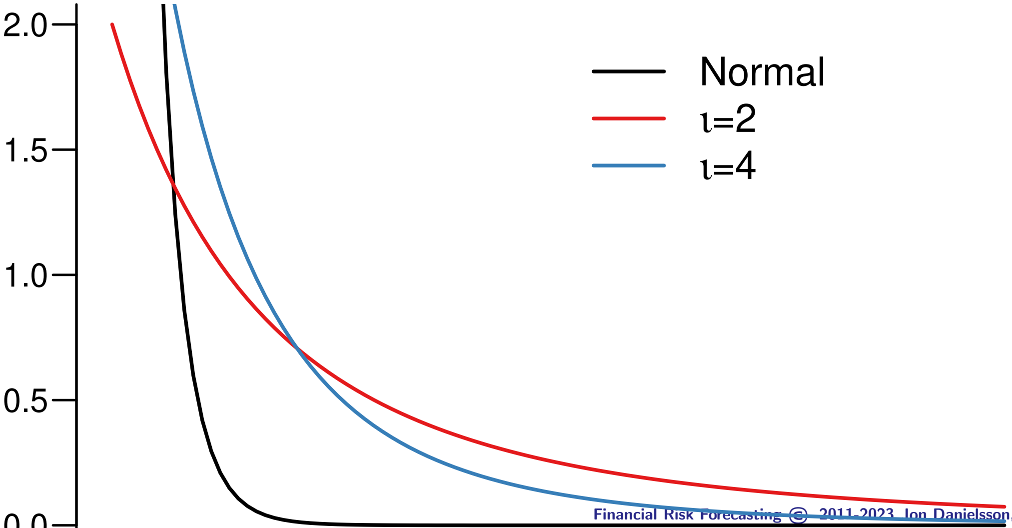
Normal and Fat Distributions

Pareto tails



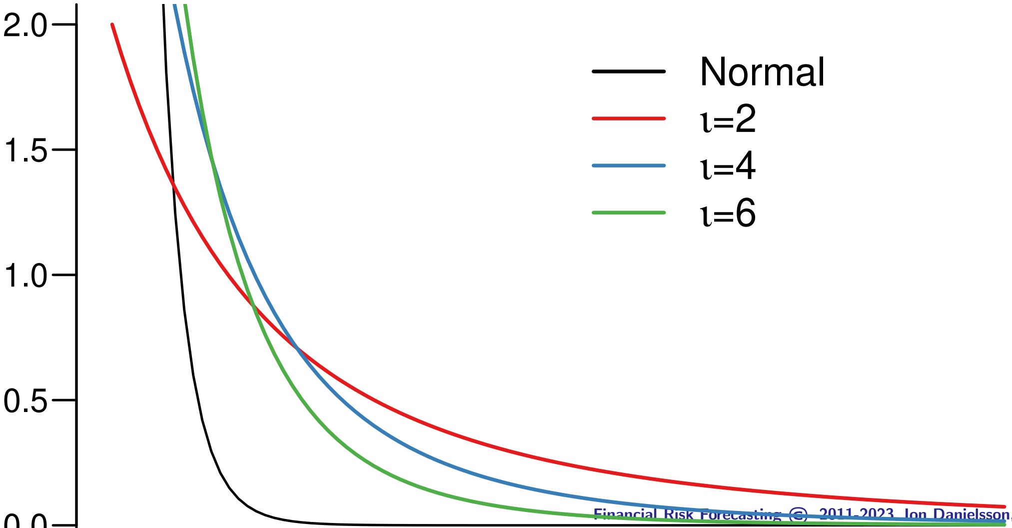
Normal and Fat Distributions

Pareto tails



Normal and Fat Distributions

Pareto tails



Normal and Fat Distributions

- The definition demonstrates that fat tails are defined by how rapidly the tails of the distribution decline as we approach infinity
- As the tails become thicker, we detect increasingly large observations that impact the calculation of moments:

$$\mathrm{E}(X^m) = \int x^m f(x) dx$$

- If $\mathrm{E}(X^m)$ exists for all positive m, such as for the normal distribution, the definition of *regular variation* implies that moments $m \geq \iota$ are not defined for fat-tailed data

Applying EVT

Implementing EVT in Practice

Two main approaches:

1. Block maxima
2. Peaks over thresholds (POT)

Block Maxima Approach

- This approach follows directly from the regular variation definition where we estimate the GEV by dividing the sample into blocks and using the maxima in each block for estimation
- The procedure is rather wasteful of data and a relatively large sample is needed for accurate estimate

Peaks Over Thresholds Approach

- This approach is generally preferred and forms the basis of our approach below
- It is based on models for all large observations that exceed a high threshold and hence makes better use of data on extreme values
- There are two common approaches to POT:
  1. Fully parametric models (eg *the Generalised Pareto distribution or GPD*)
  2. Semi-parametric models (eg *the Hill estimator*)

Generalised Pareto Distribution

- Consider a random variable X, fix a threshold u and focus on the *positive part of* $X - u$
- The distribution $F_u(x)$ is

$$F_u(x) = \Pr(X - u \leq x | X > u)$$

- If u is VaR, then $F_u(x)$ is the probability that we exceed VaR by a particular amount (a shortfall) given that VaR is violated
- Key result is that as $u \to \infty$, $F_u(x)$ converges to the GPD, $G_{\xi,\beta}(x)$

- The GPD $G_{\xi,\beta}(x)$ is

$$G_{\xi,\beta}(x) = \begin{cases} 1 - \left(1 + \xi \frac{x}{\beta}\right)^{-\frac{1}{\xi}} & \xi \neq 0 \\ 1 - \exp\left(\frac{x}{\beta}\right) & \xi = 0 \end{cases}$$

  where $\beta > 0$ is the scale parameter; $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq -\frac{\beta}{\xi}$ when $\xi < 0$
- We therefore need to estimate both shape(ξ) and scale(β) parameters when applying GDP
- Recall, for certain values of ξ the shape parameters, $G_{\xi,\beta}(.)$ becomes one of the three distributions

GEV and GPD

- The GEV is the limiting distribution of normalised maxima, whereas the GPD is the limiting distribution of normalised data beyond some high threshold
- Note, the tail index is the same for both GPD and GEV distributions
- The parameters of GEV can be estimated from the log-likelihood function of GPD

VaR Under GPD

The VaR in the GPD case is:

$$\mathrm{VaR}(p) = u + \frac{\beta}{\xi}\left[\left(\frac{1-p}{F(u)}\right)^{-\xi} - 1\right]$$

Hill Method

- Alternatively, we could use the semi-parametric Hill estimator for the tail index in distribution $F(x) \approx 1 - Ax^{-\iota}$:

$$\hat{\xi} = \frac{1}{\hat{\iota}} = \frac{1}{C_T} \sum_{i=1}^{C_T} log \frac{x_{(i)}}{u}$$

  where $x_{(i)}$ is the notation of sorted data, for example, maxima is denoted as $x_{(1)}$
- As $T \to \infty$, $C_T \to \infty$ and $C_T / T \to 0$
- Note that the Hill estimator is sensitive to the choice of threshold, u

Which Method to Choose?

- *GPD*, as the name suggests, is more general and can be applied to all three types of tails
- *Hill method* on the other hand is in the maximum domain of attraction (MDA) of the Fréchet distribution
- Hence Hill method is only valid for fat-tailed data

Risk Analysis

- After estimation of the tail index, the next step is to apply a risk measure
- The problem is finding $\text{VaR}(p)$ such that

$$\Pr\left[X \leq -\text{VaR}(p)\right] = F_X\left(-\text{VaR}(p)\right) = p$$

where $F_X(u)$ is the probability of being in the tail, that is the returns exceeding the threshold u

Risk Analysis

- Let G be the distribution of X since we are in the left tail (ie $X \leq -u$). By the Pareto assumption we have:

$$G\left(-\text{VaR}(p)\right) = \left(\frac{\text{VaR}(p)}{u}\right)^{-\iota}$$

- And by the definition of conditional probability:

$$G\left(-\text{VaR}(p)\right) = \frac{p}{F_X(u)}$$

VaR Estimator

- Equating the previous two relationship, we obtain:

$$\mathrm{VaR}(p) = u \left(\frac{F_X(u)}{p} \right)^{\frac{1}{\iota}}$$

- $F_x(u)$ can be estimated by the proportion of data beyond the threshold u, C_T/T
- The VaR estimator is therefore:

$$\widehat{\mathrm{VaR}(p)} = u \left(\frac{C_T/T}{p} \right)^{\frac{1}{\hat{\iota}}}$$

EVT Often Applied Inappropriately

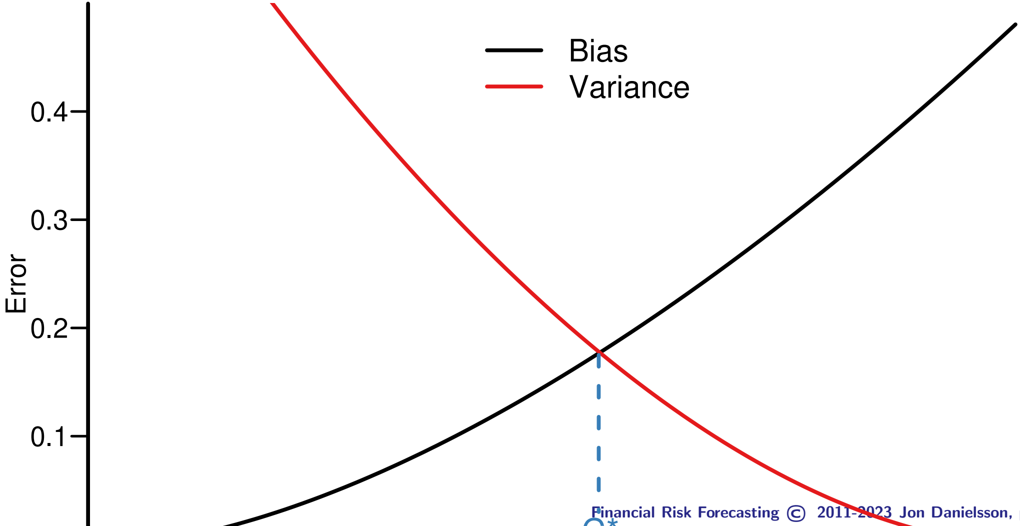
- EVT should only be applied in the tails
- The closer to the centre of the distribution, the more inaccurate the estimates are
- However, there are no rules to define when the estimates become inaccurate, it depends on the underlying distribution of the data
- In some cases, it may be accurate up to 1% or even 5%, while in other cases it is not reliable even up to 0.1%

Finding the Threshold

- Actual implementation of EVT is relatively simple and delivers good estimates where EVT holds
- The sample size T and the choice of probability level p depends on the underlying distribution of the data
- As a *rule of thumb*: $T \geq 1000$ and $p \leq 0.4\%$
- For applications with smaller sample sizes or less extreme probability levels, other techniques should be used
  - Such as HS or fat-tailed GARCH

- It can be challenging to estimate EVT parameters given the *effective sample size* is small
- This relates to choosing the number of observations in the tail, C_T
- We have 2 conflicting directions:
  1. By lowering C_T, we can reduce the estimation bias
  2. On the other hand, by increasing C_T, we can reduce the estimation variance

Optimal Threshold C_T^*

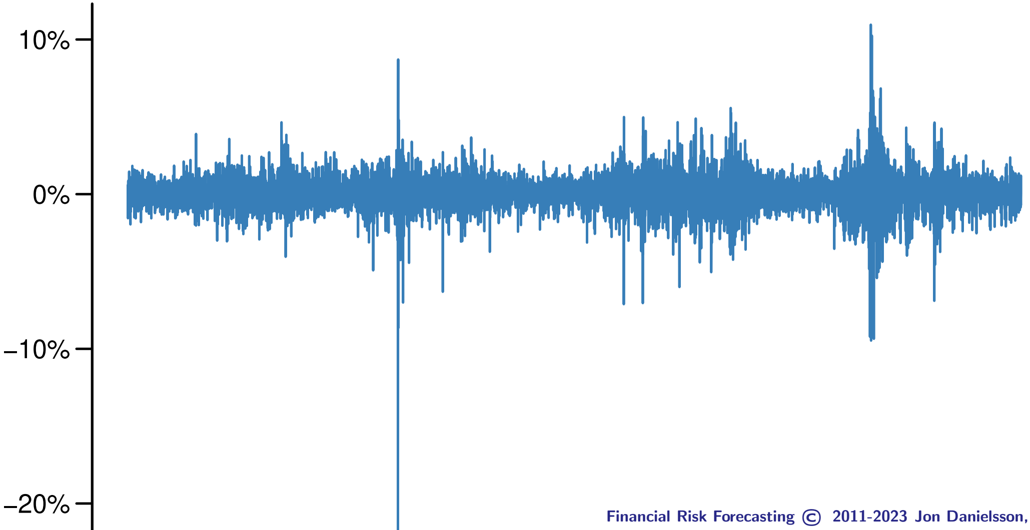


Optimal Threshold C_T^*

- If the underlying distribution is known, then deriving the optimal threshold is easy, but in such a case EVT is superfluous
- Most common approach to determine the optimal threshold is the *eyeball method* where we look for a region where the tail index seems to be stable
- More formal methods are based on minimising the mean squared error (MSE) of the Hill estimator, but such methods are not easy to implement

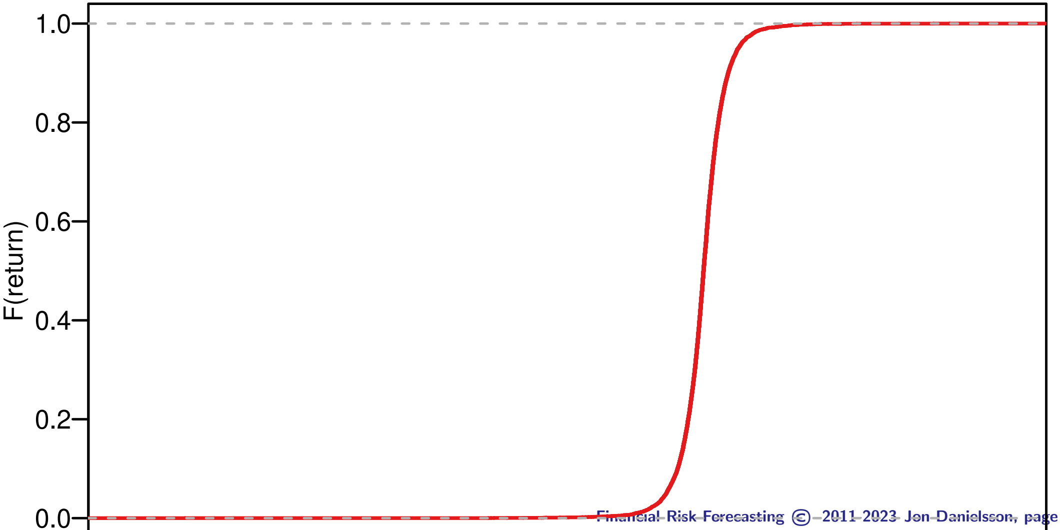
Application to the S&P-500 Index

Returns from 1975 to 2015 – 10,000 observations



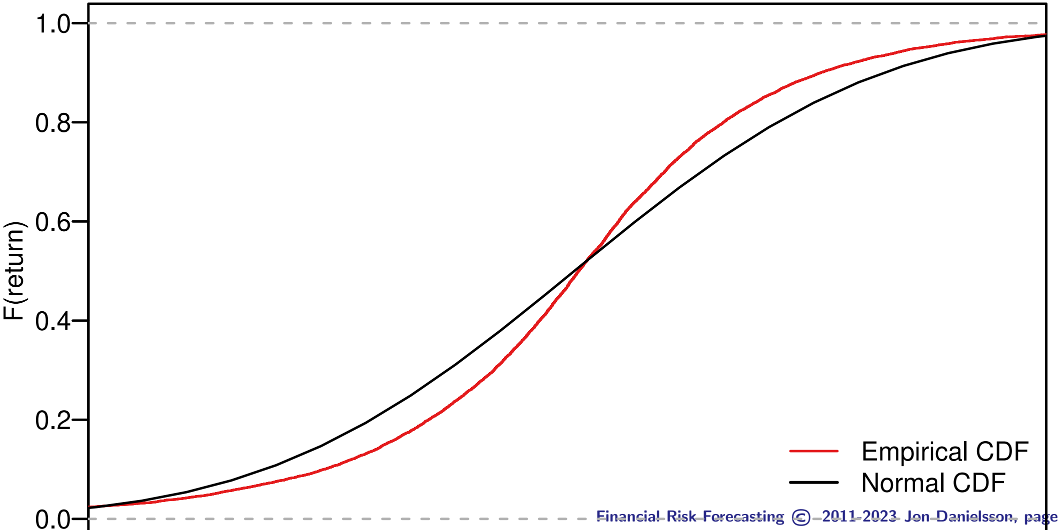
Distribution of S&P-500 Returns

Empirical distribution



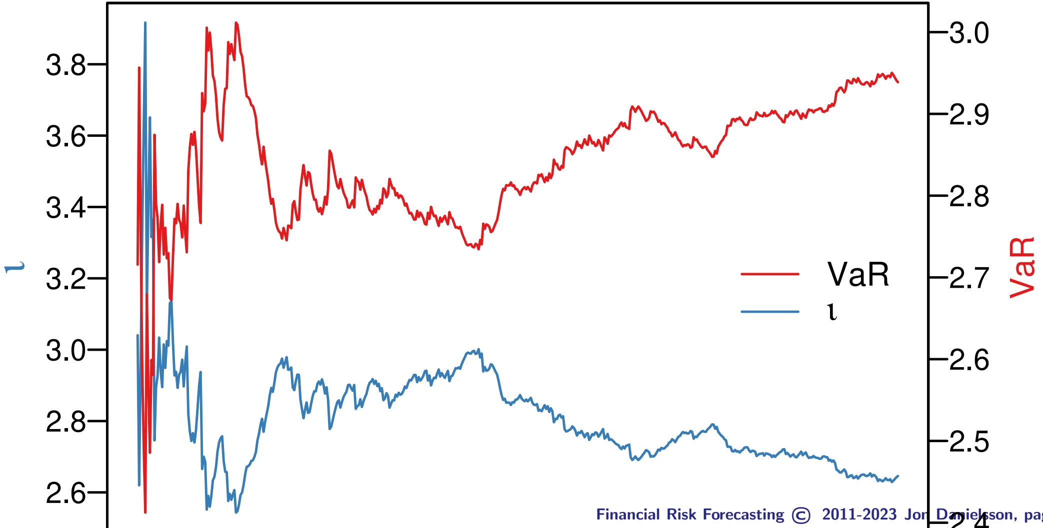
Distribution of S&P-500 Returns

Tails truncated



Hill Plot for Daily S&P-500 Returns

From 1975 to 2015



Hill Plot for Daily S&P-500 Returns

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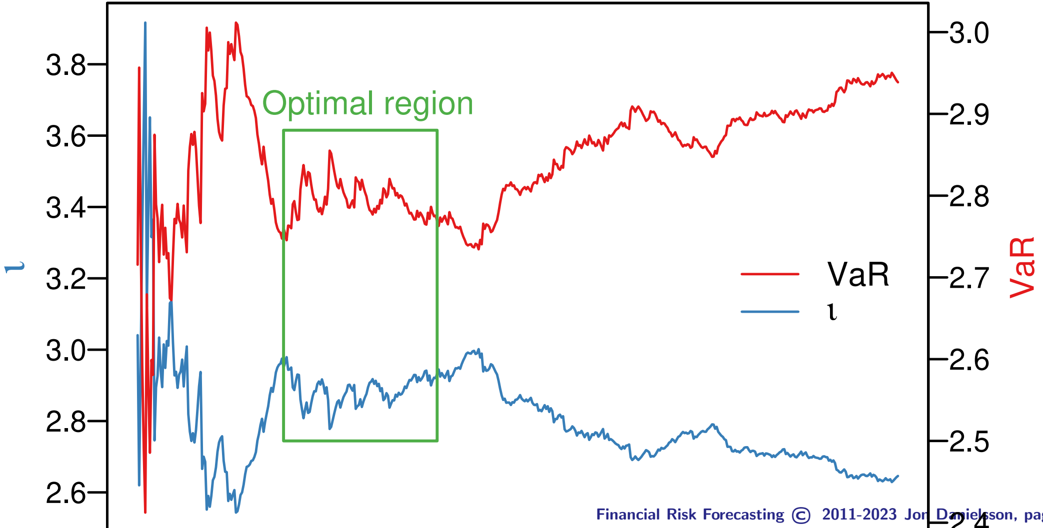
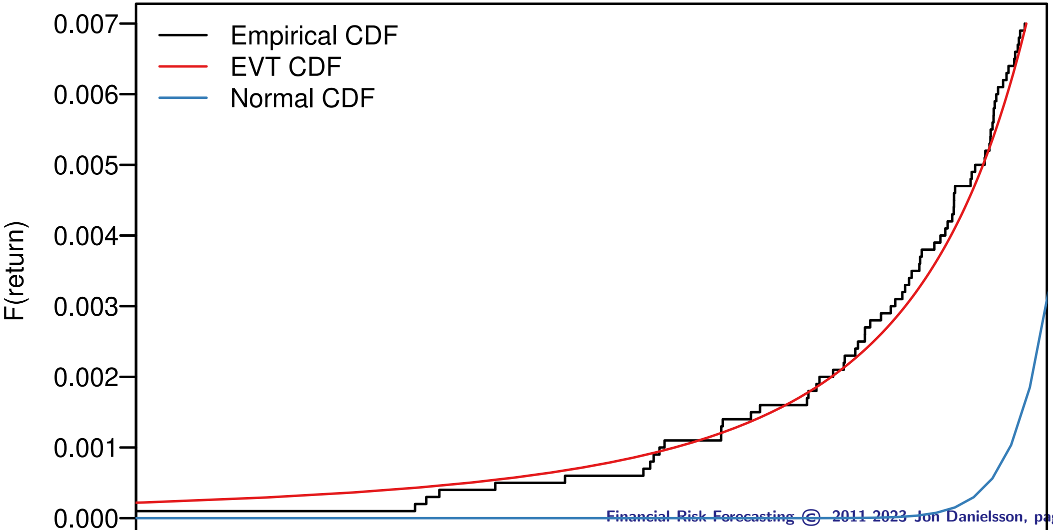


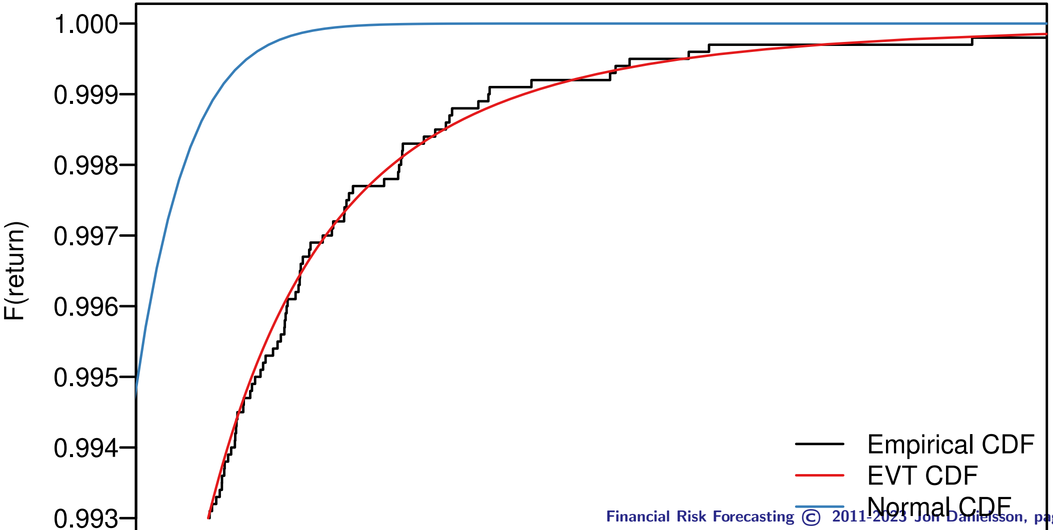
Upper and Lower Tails

The lower tail



Upper and Lower Tails

The upper tail



Aggregation and Convolution

Aggregation of Outcomes

- The act of adding up observations across time is known as *time aggregation*
- And the act of adding up observations across assets/portfolios is termed *convolution*

Feller 1971

Theorem Let X_1 and X_2 be two independent random variables with distribution functions satisfying

$$1 - F_i(x) = \mathbb{P}\{X_i > x\} \approx A_i x^{-\iota_i} \qquad i = 1, 2$$

when $x \to \infty$. Note, A_i is a constant

Then, the distribution function F of the variable $X = X_1 + X_2$ in the positive tail can be approximated by 2 cases

Case 1 When $\iota_1 = \iota_2$ we say that the random variables are first-order similar and we set $\iota = \iota_1 = \iota_2$ and F satisfies

$$1 - F(x) = \mathbb{P}\{X > x\} \approx (A_1 + A_2)x^{-\iota}$$

Case 2 When $\iota_1 \neq \iota_2$ we set $\iota = \min(\iota_1, \iota_2)$ and F satisfies

$$1 - F(x) = \mathbb{P}\{X > x\} \approx Ax^{-\iota}$$

where A is the corresponding constant

- As a consequence, if two random variables are *identically distributed*, the distribution function of the sum (Case 1) will be given by

$$\mathbb{P}\{X_1 + X_2 > x\} \approx 2Ax^{-\iota}$$

- Hence the probability doubles when we combine two observations from different days
- But if one observations comes from a fatter tailed distribution than the other, then only the heavier tail matters (Case 2)

Time Scaling

Theorem (de Vries 1998) Suppose X has finite variance with a tail index $\iota > 2$. At a constant risk level p, increasing the investment horizon from 1 to T periods increases the VaR by a factor:

$$T^{1/\iota}$$

Note, EVT distributions retain the same tail index for longer period returns

- Recall from chapter 4, under Basel Accords, financial institutions are required to calculate VaR for a 10-day holding periods
- The rules allow the 10-day VaR to be calculated by scaling the one-day VaR by $\sqrt{10}$
- The theorem shows that the scaling parameter is slower than the square-root-of-time adjustment
- Intuitively, as extreme values are more rare, they should aggregate at a slower rate than the normal distribution
- For example, if $\iota = 4$, $10^{1/\iota} = 1.78$, which is less than $\sqrt{10} = 3.16$

VaR and the Time Aggregation of Fat Tail Distributions

Risk level	5%	1%	0.5%	0.1%	0.05%	0.005%
Extreme value						
1 Day	0.9	1.5	1.7	2.5	3.0	5.1
10 Day	1.6	2.5	3.0	4.3	5.1	8.9
Normal						
1 Day	1.0	1.4	1.6	1.9	2.0	2.3
10 Day	3.2	4.5	4.9	5.9	6.3	7.5

- For one-day horizons, we see that in general EVT VaR is higher than VaR under normality, especially for more extreme risk levels
- This is balanced by the fact that 10-day EVT VaR is less than the normal VaR
- This seems to suggest that the square-root-of-time rule may be sufficiently prudent for longer horizons
- It is important to keep in mind that ι root rule (de Vries) only holds *asymptotically*

Time Dependence

Time Dependence

- Recall the assumption of IID returns in the section on EVT, which suggests that EVT may not be relevant for financial data
- Fortunately, we **do not need** an IID assumption, since EVT estimators are consistent and unbiased even in the presence of higher moment dependence
- We can explicitly model extreme dependence using the *extremal index*

Example

- Let us consider extreme dependence in a MA(1) process:

$$Y_t = X_t + \alpha X_{t-1} \qquad |\alpha| < 1$$

- Let X_t and X_{t-1} be IID such that $\Pr\{X_t > x\} \to Ax^{-\iota}$ as $x \to \infty$. Then by Feller's theorem

$$\mathbb{P}\{Y_t \geq x\} \approx (1 + \alpha^{\iota})Ax^{-\iota} \qquad \text{as } x \to \infty$$

- Dependence enters *"linearly"* by means of the coefficient α^{ι}. But the tail shape is unchanged
- This example suggest that time dependence has same effect as having an IID sample with fewer observations

- Suppose we record each observation twice:

$$Y_1 = X_1, Y_2 = X_1, Y_3 = X_2, \ldots$$

- And it increases the sample size to $D = 2T$. Let us define $M_D \equiv \max(Y_1, \ldots, Y_D)$. Evidently from Fisher-Tippet and Gnedenko theorem:

$$\mathbb{P}\{M_D \leq x\} = F^T(x) = F^{\frac{D}{2}}(x)$$

  supposing $a_T = 0$ and $b_T = 1$

- The important result here is that *dependence increases the probability that the maximum is below threshold x*

Extremal Index

Extremal index ψ It is a measure of tail dependence and $0 < \psi \leq 1$

- If the data are *independent* then we get

$$\mathbb{P}\{M_T \leq x\} \to e^{-x^{-\iota}} \qquad \text{as } T \to \infty$$

  when $a_T = 0$ and $b_T = 1$

- If the data are *dependent*, the limit distribution is

$$\mathbb{P}\{M_D \leq x\} \to \left(e^{-x^{-\iota}}\right)^{\psi} = e^{-\psi x^{-\iota}}$$

- $\frac{1}{\psi}$ is a measure of the *cluster size* in large samples, for double-recorded data $\psi = \frac{1}{2}$
- For the MA(1) process in the previous example, we obtain the following

$$\mathbb{P}\left\{T^{-\frac{1}{\iota}} M_D \leq x\right\} \to \exp\left(-\frac{1}{1+\alpha^\iota} x^{-\iota}\right)$$

where $\psi = \frac{1}{1+\alpha^\iota}$

Dependence in ARCH

- Consider the normal ARCH(1) process:

$$\begin{aligned} Y_t &= \sigma_t Z_t \\ \sigma_t^2 &= \omega + \alpha Y_{t-1}^2 \\ Z_t &\sim \mathcal{N}(0,1) \end{aligned}$$

- Subsequent returns are uncorrelated but are *not independent*, since

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_{t-1}) &= 0 \\ \mathrm{Cov}(Y_t^2, Y_{t-1}^2) &\neq 0 \end{aligned}$$

- Even when Y_t is conditionally normally distributed, we noted in chapter 2 that the unconditional distribution of Y is fat tailed
- de Haan et al. show that the unconditional distribution of Y is given by

$$\Gamma\left(\frac{\iota}{2}+\frac{1}{2}\right)=\sqrt{\pi}(2\alpha)^{-\iota/2}$$

Extremal Index for ARCH(1) – Example

- Extremal index for the ARCH(1) process can be solved using the previous equation
- From the table below, we see that the higher the α , the fatter the tails and the higher the level of clustering

α	0.10	0.50	0.90	0.99
ι	26.48	4.73	2.30	2.02
ψ	0.99	0.72	0.46	0.42

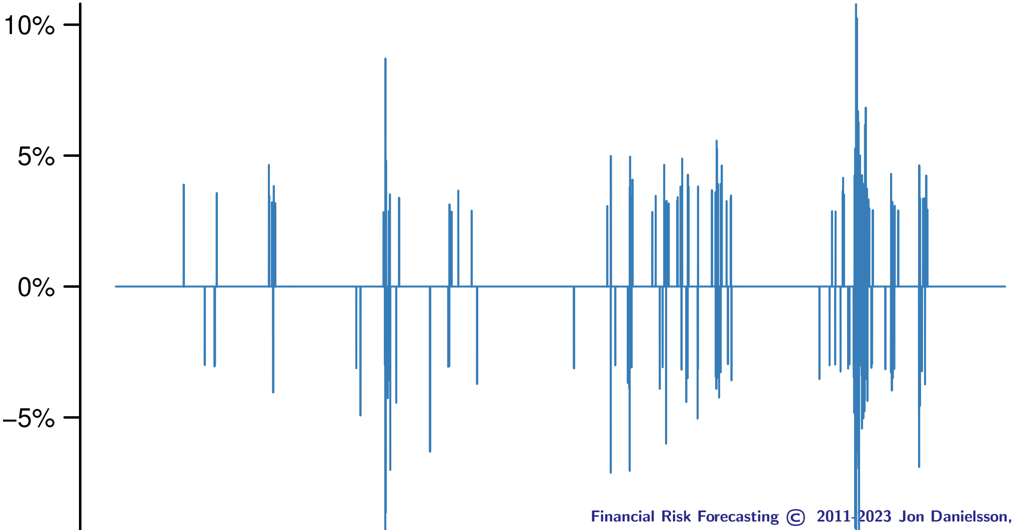
- Similar results can be obtained for GARCH

When Does Dependence Matter?

- The importance of extreme dependence and the extremal index ψ depends on the underlying applications
- Dependence can be *ignored* if we are dealing with *unconditional probabilities*
- And dependence *matters* when calculating *conditional probabilities*
- For many stochastic processes, including GARCH, the time between tail events become increasingly independent

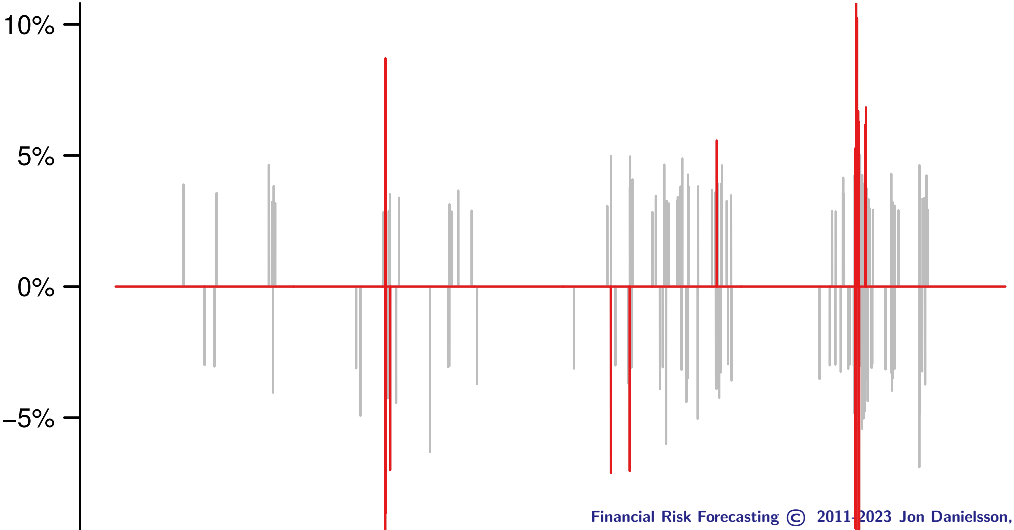
Example – S&P-500 Index Extremes

From 1970 to 2015, 1% events



Example – S&P-500 Index Extremes

From 1970 to 2015, 0.1% events



Example – S&P-500 Index Extremes

0.1% events during the crisis