Chapter 1
Financial markets, prices and risk

Jon Danielsson ©2019
London School of Economics

To accompany
Financial Risk Forecasting
www.financialriskforecasting.com
Published by Wiley 2011
Version 4.0, August 2019
The focus of this chapter

• Statistical techniques for analyzing prices and returns in financial markets
• Stock market indices, e.g. the S&P 500
• Prices, returns and volatilities
• Three stylized facts of financial returns:
  1. Volatility clusters
  2. Fat tails
  3. Nonlinear dependence
• See Appendix A for more detailed discussion on the statistical methods
• See Appendices B and C for introduction to R and Matlab
Notation

\( T \) Sample size

\( t = 1, \ldots, T \) A particular observation period (e.g. a day)

\( P_t \) Price at time \( t \)

\( R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \) Simple return

\( Y_t = \log \frac{P_t}{P_{t-1}} \) Continuously compounded return

\( y_t \) A sample realization of \( Y_t \)

\( \sigma \) Unconditional volatility

\( \sigma_t \) Conditional volatility

\( K \) Number of assets

\( w \) \( K \times 1 \) vector of portfolio weights

\( \nu \) Degrees of freedom of the Student-\( t \)

\( \iota \) Tail index

\( d \) dividends
Risk is latent

- A farmer feeds his turkeys every day
- A scientist turkey discovers:
  - “Food arrives every morning at 7am”
- And tells this to the other turkeys
- On Thanksgiving morning
- But food doesn’t arrive
- Instead, the farmer kills all the turkeys
<table>
<thead>
<tr>
<th>Prices &amp; returns</th>
<th>Data/Code</th>
<th>Volatility</th>
<th>Fat tails</th>
<th>NLD</th>
<th>Issues</th>
<th>Copulas</th>
</tr>
</thead>
</table>

**Prices, returns and indices**
Total returns 1900-2016
USA

Equities and Bonds return comparison from 1900 to 2016 for the USA.
Total returns 1900-2016

- Equities
- Bonds

Countries: South Africa, Australia, USA, New Zealand, Sweden, Canada, Denmark, Finland, UK, Netherlands, Switzerland, Ireland, Spain, Portugal, France, Belgium, Austria, USA, Canada, New Zealand, UK, Netherlands, Switzerland, Ireland, Spain, Portugal, France, Belgium, Austria, USA, Canada, New Zealand, UK, Netherlands, Switzerland, Ireland, Spain, Portugal, France, Belgium, Austria

Y-axis: 0.01, 0.1, 1, 10, 100, 1000

Stock indices

- A *stock market index* shows how a representative portfolio of stock prices changes over time.
- A *price-weighted* index weighs stocks based on their prices.
  - A stock trading at $100 makes up 10 times more of total than a stock trading at $10.
- A *value-weighted* index weighs stocks according to the total market value of their outstanding shares.
  - Impact of change in stock price proportional to overall market value.
Stock indices

- The most widely used index is the Standard & Poor’s 500 (S&P 500) — largest 500 traded companies in the US
- Examples of value-weighted indices:
  - S&P 500, FTSE 100 (UK), TOPIX (Japan)
- Examples of price-weighted indices:
  - Dow Jones Industrial Average (US), Nikkei 225 (Japan)
Prices and returns

- Denote prices by $P_t$. Usually we are more interested in the *return* we make on an investment.

**Definition Return** The relative change in the price of a financial asset over a given time interval, often expressed as a percentage.

- There are two types of returns:
  1. *Simple* $(R)$
  2. *Compound* $(Y)$
Simple returns

**Definition** A simple return is the percentage change in prices

\[ R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \]

- Including dividends

\[ R_t = \frac{P_t - P_{t-1} + d}{P_{t-1}} \]
Continuously compounded returns

**Definition** The logarithm of gross return

\[ Y_t = \log(1 + R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}) \]

\[ P_{t+1} = P_t e^R \]
Simple and continuous

- The difference between $R_t$ and $Y_t$ is not large for daily returns.
- As the time between observations goes to zero, so does the difference between the two measures:

$$\lim_{\Delta t \to 0} Y_t = R_t$$

$$\log(1000) - \log(990) = 0.01005 \approx \frac{1000}{990} - 1 = 0.0101$$

$$\log(1000) - \log(800) = 0.223 \neq \frac{1000}{800} - 1 = 0.25$$
Symmetry

- Continuous returns are \( \textit{symmetric} \)

\[
\log \left( \frac{1000}{200} \right) = - \log \left( \frac{200}{1000} \right)
\]

- Simple are not

\[
\frac{1000}{200} - 1 \neq - \left( \frac{200}{1000} - 1 \right)
\]
• Simple returns are
  • Used for accounting purposes
  • Investors are usually concerned with simple returns

• Continuously compounded returns have some advantages
  • Mathematics is easier (e.g. how returns aggregate over many periods, used in Chapter 4)
  • Used in derivatives pricing, e.g. the Black–Scholes model
**Issues for portfolios**

- $R_{t,\text{portfolio}}$ return on a portfolio

- Weighted sum of returns of individual assets:

  $$R_{t,\text{portfolio}} = \sum_{k=1}^{K} w_k R_{t,k} = w' R_t$$

- While

  $$Y_{t,\text{portfolio}} = \log \left( \frac{P_{t,\text{portfolio}}}{P_{t-1,\text{portfolio}}} \right) \neq \sum_{k=1}^{K} w_k \log \left( \frac{P_{t,k}}{P_{t-1,k}} \right)$$

- Because the log of a sum does not equal the sum of logs
Issues

- Stock prices can give misleading performance indication
- For more accuracy, adjust returns for dividends, splits and buybacks
- *Total returns* adjust for all of these
- If you only adjust for number of shares, we get what is sometimes called *adjusted returns*
S&P 500 index

- US Civil War
- Great Depression
- Nifty-fifty
- Internet Bubble
- 2007 crisis
S&P 500 returns

US Civil War
Great Depression
Asian crisis
1987 crash
2008 crisis
S&P 500 statistics
1929 to 2016, daily returns

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0274%</td>
</tr>
<tr>
<td>Standard error</td>
<td>1.143%</td>
</tr>
<tr>
<td>Min</td>
<td>−20.47%</td>
</tr>
<tr>
<td>Max</td>
<td>16.60%</td>
</tr>
<tr>
<td>Skewness</td>
<td>−0.096</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>20.31</td>
</tr>
</tbody>
</table>

- Note how small mean is compared to the s.e. (volatility)
- But mean grows at rate $T$ and the volatility at $\sqrt{T}$
Three stylized facts

Present in most financial returns

Volatility clusters
Fat tails
Nonlinear dependence
Software

• Excel is useless for we are trying to do here

• Four main software choices
  1. Julia
  2. Python (Numpy)
  3. R, what we use in this course
  4. Matlab

• Go to
Data

- Financial data can be obtained from many sources, e.g.
  1. wrds.wharton.upenn.edu (CRSP)
  2. Bloomberg
  3. finance.yahoo.com (often best but does not always work)
- Best to save data as a CSV file and import that into R/Matlab
- Can also use Excel files, but that is a bit more clumsy. CSV is in text format so can easily look at data
- Can download data directly from databases or internet, but that depends on access, and is often slow, so better to save locally
- Get SP-500 CSV file from www.financialriskforecasting.com/data/sp-500.csv
Representing data

- To read data into R use `read.csv`
- That returns a *data frame* not a *matrix*
- For matrices, all the columns are the same type (numbers or text)
- While a data frame can have some columns with text and others numbers
- By default, the first row in a CSV file provides the column names
- And by default, the rows have a number 1, 2, 3, ...
- It is easy to change the row names and column names
R

```
sp500 = read.csv("sp−500.csv")
sp500 = read.csv("https://www.financialriskforecasting.com/data/sp−500.csv")
names(sp500)
[1] "date"  "price"
head(sp500)
  date   price
1 20000103 1455.22
2 20000104 1399.42
3 20000105 1402.11
> names(sp500)
[1] "date"  "price"
```
R

```r
plot(sp500[,2], type='l')
plot(sp500$price)

y = diff(log(sp500[,2]))
plot(y, type='l')
mean(y)
sd(y)
```
\[
\text{sp500}\$y = \text{diff}(\log(\text{sp500}[\ ,2]))
\]

Error in `$\leftarrow .data.frame`(`*tmp*`‘, y, value = c( : 
replacement has 4275 rows, data has 4276

\[
\text{sp500}\$y = \text{c}(\text{NA}, \text{diff}(\log(\text{sp500}[\ ,2])))
\]

\[
> \text{head(}\text{sp500})
\]

<table>
<thead>
<tr>
<th>date</th>
<th>price</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 20000103</td>
<td>1455.22</td>
<td>NA</td>
</tr>
<tr>
<td>2 20000104</td>
<td>1399.42</td>
<td>-0.0390992269</td>
</tr>
<tr>
<td>3 20000105</td>
<td>1402.11</td>
<td>0.0019203798</td>
</tr>
</tbody>
</table>
A vector of stock prices can be indexed by numbers 1, 2, 3, ...

Or by date, and even time. 1/1/2019, 2/1/2019, 09:24 3/1/2019

It is usually best to avoid using dates and times

But when making pictures, and some other operations, useful

In R, the library zoo provides date and time facilities

And library lubridate is useful for conversion of dates in text format into something R knows is a date
R timeseries

```r
# install.packages("zoo")
# install.packages("lubridate")

library(zoo)
library(lubridate)

p=zoo(sp500$price, order.by=ymd(sp500$date))
plot(p)
```
Saving data

- Any data in R can be saved and read in later.

```r
save(sp500, file="sp500.RData")
load("sp500.RData")
```
Volatility
Volatility

The standard deviation/error of returns

- Two concepts of volatility:
  - *Unconditional volatility* is volatility over an entire time period \((\sigma)\)
  - *Conditional volatility* is volatility in a given time period, conditional on what happened before \((\sigma_t)\)
- Clear evidence of cyclical patterns in volatility over time, both in the short run and the long run
Calculations

- Daily volatility

\[
\sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (y_i - \mu)^2}
\]

- Annualised

\[
\sqrt{250} \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (y_i - \mu)^2}
\]

R

\[
\text{sd}(y) \quad \text{sqrt}(250) * \text{sd}(y)
\]
Cycles in volatility — SP-500

- 1920’s: 35%
- 1940’s: 30%
- 1960’s: 10%
- 1980’s: 25%
- 2000’s: 20%
Cycles in volatility — SP-500

![Graph showing cycles in volatility for the SP-500 index from 1980 to 2015. The graph includes bars representing volatility levels, with peaks in 1987, 1990, 2000, 2008, and 2015. The red line indicates the mean volatility level.](image-url)
Cycles in volatility — SP-500

The graph shows the volatility of the SP-500 index from 2000 to 2015. The volatility is measured in percentages, with the graph depicting fluctuations around the mean level. There are noticeable peaks in 2008 and 2010, indicating higher volatility during these times compared to other periods.
Simulated volatility clusters
Volatility clusters

- Volatility changes over time in a way that is partially predictable
- *Volatility clusters*
- Engle (1982) suggested a way to model this phenomenon
  - His autoregressive conditional heteroskedasticity (ARCH) model is discussed in Chapter 2
Autocorrelations

• Correlations measure how 2 variables \((x, y)\) move together

\[
\text{Corr}(x, y) = \frac{1}{N - 2} \sum_{i=1}^{N} (x - \mu_x)(y - \mu_y)
\]

• Autocorrelations measure how a single variable is correlated with itself
  • 1 lag
    \[
    \hat{\beta}_1 = \text{Corr}(x_1, \ldots, N-1, x_2, \ldots, N)
    \]
  • \(N\) lags
    \[
    \hat{\beta}_i = \text{Corr}(x_1, \ldots, N-i, x_{i+1}, \ldots, N)
    \]

\[\text{acf}(y, 20)\]
• If autocorrelations are statistically significant there is evidence for predictability
• The coefficients of an autocorrelation function (ACF) give the correlation between observations and lags
• We will test both returns ($y$), predictability in mean (price forecasting or alpha)
• And returns squared ($y^2$), predictability in volatility
The LB test for autocorrelations

- Joint significance of autocorrelation coefficients \((\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_N)\) can be tested by using the Ljung-Box (LB) test

\[
J_N = T(T + 2) \sum_{i=1}^{N} \frac{\hat{\beta}_i^2}{T - N} \sim \chi^2(N)
\]

R

```
Box.test(y, lag = 20, type = c("Ljung-Box"))
```
S&P 500 1929 to 2015 ACF of daily returns
S&P 500 1929 to 2015 ACF of squared daily returns
### LB tests for S&P 500

#### Daily returns

<table>
<thead>
<tr>
<th>$N$</th>
<th>LB statistic, 21 lags</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>22,752</td>
<td>95.9</td>
<td>$1.527 \times 10^{-11}$</td>
</tr>
<tr>
<td>2,500</td>
<td>185.2</td>
<td>$&lt; 2.2 \times 10^{-16}$</td>
</tr>
<tr>
<td>100</td>
<td>18.7</td>
<td>0.606</td>
</tr>
</tbody>
</table>

#### Daily returns squared

<table>
<thead>
<tr>
<th>$T$</th>
<th>LB statistic, 21 lags</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>22,752</td>
<td>12,633.0</td>
<td>$&lt; 2.2 \times 10^{-16}$</td>
</tr>
<tr>
<td>2,500</td>
<td>4,702.1</td>
<td>$&lt; 2.2 \times 10^{-16}$</td>
</tr>
<tr>
<td>100</td>
<td>46.0</td>
<td>0.00129</td>
</tr>
</tbody>
</table>
Fat tails
Definition Fat tails A random variable is said to have fat tails if it exhibits more extreme outcomes than a normally distributed random variable with the same mean and variance.

- The mean–variance model assumes normality.
Fat tails

- The tails are the extreme left and right parts of a distribution
- If the tails are fat, there is a higher probability of extreme outcomes than one would get from the normal distribution with the same mean and variance
- Also implies that there is a lower probability of non-extreme outcomes
- Probabilities are between zero and one so the area under the distribution is one
The Student–t distribution

- The degrees of freedom, \( \nu \), of the Student–t distribution indicate how fat the tails are
- \( \nu = \infty \) implies the normal
- \( \nu < 2 \) superfat tails
- For a typical stock \( 3 < \nu < 5 \)
- The Student–t is convenient when we need a fat tailed distribution
Tails

probability

outcome

−3  −2  −1   0   1   2   3

−3 −2 −1 0 1 2 3

0.0 0.1 0.2 0.3 0.4
Tails

outcome

probability

-3 -2 -1 0 1 2 3

-0.0 -0.1 -0.2 -0.3 -0.4

−3 −2 −1 0 1 2 3

0.0

0.1

0.2

0.3

0.4

Tails

- normal
- Student-\(t(2)\)
Tails

- normal
- Student–t(2)
- chi–2

probability

outcome

-3 −2 −1 0 1 2 3
Tails Zoom

- normal
- Student-t(2)
- chi-2

Outcome

Probability

-4.0 -3.5 -3.0 -2.5 -2.0 -1.5
Probability of extreme outcomes

• If S&P 500 returns were normally distributed, the probability of a one-day drop of 23% would be $5.51 \times 10^{-89}$!

• The table below gives probabilities of different returns assuming normality

<table>
<thead>
<tr>
<th>Returns above or below</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>0.385</td>
</tr>
<tr>
<td>2%</td>
<td>0.0820</td>
</tr>
<tr>
<td>3%</td>
<td>0.00909</td>
</tr>
<tr>
<td>5%</td>
<td>$1.37 \times 10^{-5}$</td>
</tr>
<tr>
<td>15%</td>
<td>$6.92 \times 10^{-39}$</td>
</tr>
<tr>
<td>23%</td>
<td>$5.51 \times 10^{-89}$</td>
</tr>
</tbody>
</table>
Max and min of S&P 500 returns

Per decade, daily returns

-20 %
-10 %
0 %
10 %
20 %

20's
30's
40's
50's
60's
70's
80's
90's
00's
10's
Max and min of S&P 500 returns

Per year, daily returns

- 1986
- 1988
- 1990
- 1992
- 1994
- 1996
- 1998
- 2000
- 2002
- 2004
- 2006
- 2008
- 2010
- 2012
- 2014
Empirical density vs. normal

S&P 500 daily returns, 2000 to 2015

Density

Outcomes
Empirical density vs. normal

S&P 500 daily returns, 2000 to 2015

Cumulative probability vs. outcomes

- Data higher than normal
- Data lower
- Data higher
- Data lower than normal

Returns vs. Normal
Empirical density vs. normal

S&P 500 daily returns, 2000 to 2015

Cumulative probability

Outcomes

Returns

Normal

Prices & returns  Data/Code  Volatility  Fat tails  NLD  Issues  Copulas
Non–normality and fat tails

• Three observations:
  1. Peak is higher than normal
  2. Sides are lower than normal
  3. Tails are much thicker (fatter) than normal
Identification of fat tails

- Two main approaches for identifying and analyzing tails of financial returns: statistical tests and graphical methods.
- The *Jarque-Bera* (JB) and the *Kolmogorov-Smirnov* (KS) tests can be used to test for fat tails.
- *QQ plots* allow us to analyze tails graphically by comparing quantiles of sample data with quantiles of reference distribution.
- An alternative graphical method for detecting fat tails is plotting *sequential moments*. 
Jarque–Bera test

- The Jarque–Bera (JB) test is a test for normality and may point to fat tails if rejected.
- The JB test statistic is
  \[
  \frac{T}{6} \text{Skewness}^2 + \frac{T}{24} (\text{Kurtosis} - 3)^2 \sim \chi^2(2)
  \]

R

```r
# install.packages("tseries")
library(tseries)
jarque.bera.test(y)
```
Kolmogorov-Smirnov test

- Based on minimum distance estimation comparing sample with a reference distribution, like the normal
QQ plots

- A QQ plot (quantile-quantile plot) compares the quantiles of sample data against quantiles of reference distribution.
- Used to assess whether a set of observations has a particular distribution.
- Can also be used to determine whether two datasets have the same distribution.

R

```r
library(car)
qqPlot(y)
qqPlot(y, distribution="t", df=5)
```
Daily S&P 500 returns vs. Normal

QQ plot, 1989 to 2015

Theoretical quantiles

Sample quantiles

more extremes than normal

more extremes than normal
Daily S&P 500 returns vs. Normal

- Many observations seem to deviate from normality and the QQ-plot has clear S shape
- Indicates that returns have fatter tails than normal, but how much fatter?
- We can use Student-\(t\) with different degrees of freedom as reference distribution (fewer degrees of freedom give fatter tails)
Daily S&P 500 returns vs. Student t(5)
Daily S&P 500 returns vs. Student t(4)
Daily S&P 500 returns vs. Student t(3)
Nonlinear dependence
Correlations

- Correlations are a linear concept
  \[ y = \alpha x + \epsilon \]
- Then \( \alpha \) is proportional to the correlation between \( x \) and \( y \)
- A different way to say that is \textit{linear dependence}
- The relationship between the two variables is always the same regardless of the magnitude of the variables
- Under the normal distribution, dependence is linear
- Key assumption for the mean–variance model
Nonlinear dependence

- Nonlinear dependence (NLD) implies that dependence between variables changes depending on some factor. In finance, perhaps according to market conditions
  - Example: Different returns are relatively independent during normal times, but highly dependent during crises
- If returns were jointly normal, correlations would decrease for extreme events, but empirical evidence shows exactly the opposite
- Assumption of linear dependence does not hold in general
Evidence of nonlinear dependence

Daily returns for Microsoft, Morgan Stanley, Goldman Sachs and Citigroup

### May 5, 1999 - June 12, 2015

<table>
<thead>
<tr>
<th></th>
<th>MSFT</th>
<th>MS</th>
<th>GS</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS</td>
<td>46%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GS</td>
<td>46%</td>
<td>81%</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>37%</td>
<td>65%</td>
<td>63%</td>
</tr>
</tbody>
</table>

### August 1, 2007 - August 15, 2007

<table>
<thead>
<tr>
<th></th>
<th>MSFT</th>
<th>MS</th>
<th>GS</th>
</tr>
</thead>
<tbody>
<tr>
<td>MS</td>
<td>93%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GS</td>
<td>82%</td>
<td>94%</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>87%</td>
<td>93%</td>
<td>92%</td>
</tr>
</tbody>
</table>
Implications of fat tails

• Non–normality and fat tails have important consequences in finance
• Assumption of normality may lead to a gross underestimation of risk
• However, the use of non-normal techniques is highly complicated, and unless correctly used, may lead to incorrect outcomes
Volatility and fat tails

- Volatility is the correct measure of risk if and only if the returns are normal.
- If they follow the Student-t or any of the fats, then volatility will only be partially correct as a risk measure.
- We discuss this in more detail in Chapter 4.
The quant crisis of 2007

- Many hedge funds using quantitative trading strategies ran into serious difficulties in June 2007
- The correlations in their assets increased very sharply
- So they were unable to get rid of risk
Goldman Sachs’s flagship Global Alpha fund (summer of 2007)

“We were seeing things that were 25–standard deviation moves, several days in a row,” said David Viniar, Goldman’s chief financial officer. “There have been issues in some of the other quantitative spaces. But nothing like what we saw last week.”
Lehmans (summer of 2007)

“Wednesday is the type of day people will remember in quantland for a very long time,” said Mr. Rothman, a University of Chicago Ph.D. who ran a quantitative fund before joining Lehman Brothers. “Events that models only predicted would happen once in 10,000 years happened every day for three days.”
Volatility and fat tails

- Goldman’s 25 sigma event under the normal has a probability of $3 \times 10^{-138}$
- Age of the universe is estimated to be $5 \times 10^{12}$ days while the earth is $1.6 \times 10^{12}$ days old
- Goldman expected to suffer a one–day loss of this magnitude less than one every $1.5 \times 10^{125}$ universes
- Or perhaps the distributions were really not Gaussian
Diversification and fat tails

• Suppose you go to a dodgy buffet restaurant
• Where you worry about food poisoning in one of the foods offered
• But you don’t know which
• And are really hungry
• How many different types of food do you try?
Copulas
Exceedance correlations

- Exceedance correlations show the correlations of (standardized) stock returns $X$ and $Y$ as being conditional on exceeding some threshold, i.e.

$$\tilde{\rho}(p) = \begin{cases} \text{Corr}[X, Y|X \leq Q_X(p) \text{ and } Y \leq Q_Y(p)], & \text{for } p \leq 0.5 \\ \text{Corr}[X, Y|X > Q_X(p) \text{ and } Y > Q_Y(p)], & \text{for } p > 0.5 \end{cases}$$

where $Q_X(p)$ and $Q_Y(p)$ are the $p$-th quantiles of $X$ and $Y$ given a distributional assumption.

- Can be used to detect NLD.
**Exceedance plot**

**Bivariate normal and Student-t**

- Normal $\rho=0.5$
- Normal $\rho=0.7$
- Student-t(3) $\rho=0.5$
Empirical exceedance plot
Disney and IBM daily returns, January 1986 to June 2015

Data
Normal
Student−t(3)
Copulas and nonlinear dependence

• How do we model nonlinear dependence more formally?
• One approach is multivariate volatility models (see Chapter 3)
• Alternatively we can use copulas, which allow us to create multivariate distributions with a range of types of dependence
Intuition behind copulas

- A copula is a convenient way to obtain the dependence structure between two or more random variables, taking NLD into account.
- We start with the marginal distributions of each random variable and end up with a copula function.
- The copula function joins the random variables into a single multivariate distribution by using their correlations.
Intuition behind copulas

- The random variables are transformed to uniform distributions using the *probability integral transformation*
- The copula models the dependence structure between these uniforms
- Since the probability integral transform is invertible, the copula also describes the dependence between the original random variables
Theory of copulas

• Suppose $X$ and $Y$ are two random variables representing returns of two different stocks, with densities $f$ and $g$:

$$X \sim f \text{ and } Y \sim g$$

• Together, the joint distribution and marginal distributions are represented by the joint density $h$:

$$(X, Y) \sim h$$

• We focus separately on the marginal distributions $(F, G)$ and the copula function $C$, which combines them into the joint distribution $H$
Theory of copulas

- We want to transform $X$ and $Y$ into random variables that are distributed uniformly between 0 and 1, removing individual information from the bivariate density $h$

**Theorem 1.1** Let a random variable $X$ have a continuous distribution $F$, and define a new random variable $U$ as:

$$U = F(X)$$

Then, regardless of the original distribution $F$:

$$U \sim \text{Uniform}(0, 1)$$
Theory of copulas

• Applying this transformation to $X$ and $Y$ we obtain:

$$U = F(X) \text{ and } V = G(Y)$$

• Using this we arrive at the following theorem

**Theorem 1.2** Let $F$ be the distribution of $X$, $G$ the distribution of $Y$ and $H$ the joint distribution of $(X,Y)$. Assume that $F$ and $G$ are continuous. Then there exists a unique copula $C$ such that:

$$H(X, Y) = C(F(X), G(Y))$$
Theory of copulas

- In applications we are more likely to use densities:

\[ h(X, Y) = f(X) \times g(Y) \times C(F(X), G(Y)) \]

- The copula contains all dependence information in the original density \( h \), but none of the individual information.

- Note that we can construct a joint distribution from any two marginal distributions and any copula, and we can also extract the implied copula and marginal distributions from any joint distribution.
The Gaussian copula

- One example of a copula is the *Gaussian copula*
- Let $\Phi(\cdot)$ denote the normal (Gaussian) distribution and $\Phi^{-1}(\cdot)$ its inverse
- Let $U, V \in [0, 1]$ be uniform random variables and $\Phi_\rho(\cdot)$ the bivariate normal with correlation coefficient $\rho$
- Then the Gaussian copula function can be written as:

$$C(U, V) = \Phi_\rho(\Phi^{-1}(U), \Phi^{-1}(V))$$

- This function allows us to join the two marginal distributions into a single bivariate distribution
Application of copulas

- To illustrate we use the same data on Disney and IBM as used before.
- By comparing a scatterplot for simulated bivariate normal data with one for the empirical data, we see that the two do not have the same joint extremes.
Gaussian scatterplot

Asset 2

-20
-10
0
10

Asset 1

-30
-20
-10
0
10
Empirical scatterplot

Daily Disney and IBM returns, January 1986 to June 2015

Both crash together
Application of copulas

- We estimate two copulas for the data, a Gaussian copula and a Student-\(t\) copula
- The copulas can be drawn in three dimensions
Fitted Gaussian copula

Daily Disney and IBM returns, January 1986 to June 2015
Fitted Student-t copula

Daily Disney and IBM returns, January 1986 to June 2015
Application of copulas

- It can be difficult to compare distributions by looking at three-dimensional graphs
- Contour plots may give a better comparison
Contours of Gaussian copula

Daily Disney and IBM returns, January 1986 to June 2015
Contours of Student-t copula

Daily Disney and IBM returns, January 1986 to June 2015
Clayton’s copula

- As noted earlier, there are a number of copulas available
- One widely used is the Clayton copula, which allows for asymmetric dependence
- Parameter $\theta$ measures the strength of dependence
- We estimate a Clayton copula for the same data as before
Contours of Clayton's copula, $\theta = 1$
Contours of Clayton’s copula, $\theta = 0.483$

Daily Disney and IBM returns, January 1986 to June 2015