Financial Risk Forecasting

Chapter 3

Multivariate volatility models

Jon Danielsson ©2019
London School of Economics

To accompany
Financial Risk Forecasting
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Multivariate volatility

EWMA

CCC

DCC

Large problems

Go-GARCH

Estimation comparison

BEKK
The previous chapter focused on the volatility of a single asset
In most we hold a portfolio of assets
And therefore need to estimate both the volatility of each asset in the portfolio
And the correlations between all the assets
This means that it is much more complicated to estimate multivariate volatility models than univariate models
The focus of this chapter is on

- EWMA
- Orthogonal GARCH
- CCC and DCC models
- Estimation comparison
- Multivariate extensions of GARCH (MV GARCH and BEKK)
### Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$\Sigma_t$</td>
<td>Conditional covariance matrix</td>
</tr>
<tr>
<td>$Y_{t,k}$</td>
<td>Return on asset $k$ at time $t$</td>
</tr>
<tr>
<td>$y_{t,k}$</td>
<td>Sample return on asset $k$ at time $t$</td>
</tr>
<tr>
<td>$\mathbf{y}<em>t = {y</em>{t,k}}$</td>
<td>Vector of sample returns on all assets at time $t$</td>
</tr>
<tr>
<td>$\mathbf{y} = {y_t}$</td>
<td>Matrix of sample returns on all assets and dates</td>
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<td>$A$ and $B$</td>
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<td>$R$</td>
<td>Correlation matrix</td>
</tr>
<tr>
<td>$D_t$</td>
<td>Conditional variance forecast</td>
</tr>
</tbody>
</table>
Data

• Amazon and Google daily stock price from 2006-2015
  www.financialriskforecasting.com/data/amzn-goog.csv

```r
p <- read.csv("http://www.financialriskforecasting.com/data/amzn-goog.csv")
y <- apply(log(p[, 2:3]), 2, diff)
print(head(y))
```
R estimation

- It is easy to implement EWMA directly in R
- But no package exists for estimating all the models discussed below
- We mostly use Alexios Ghalanos’s rmgarch
Matlab estimation

- It is easy to implement EWMA directly in Matlab
- For the other models it is generally best to use some library functions
- The only one I know of is Kevin Sheppard’s MFE toolbox www.kevin/sheppard.com/MFE_Toolbox
- His documentation lags behind the code and does not mention the multivariate volatility functions
- But if you download the toolbox you can see his code and each function is documented at the top
Other reading

- Many surveys, e.g.


- Multivariate GARCH models for large-scale applications: A survey,
  Kris Boudt, Alexios Galanos, Scott Payseur, Eric Zivot

- Handbook of Statistics,

- Elsevier,

- 2019,
Multivariate Volatility Forecasting
• Consider the univariate volatility model:

\[ Y_t = \sigma_t Z_t \]

where \( Y_t \) are returns, \( \sigma_t \) is conditional volatility and \( Z_t \) are random shocks.

• If there are \( K > 1 \) assets under consideration, it is necessary to indicate which asset and parameters are being referred to, so the notation becomes more cluttered:

\[ Y_{t,i} = \sigma_{t,i} Z_{t,i} \]

where the first subscript indicates the date and the second subscript the asset.
Conditional covariance matrix $\Sigma_t$

- The conditional covariance between two assets $i$ and $j$ is indicated by:
  \[
  \text{Cov}(Y_{t,i}, Y_{t,j}) \equiv \sigma_{t,ij}
  \]
- In the three-asset case (note that $\sigma_{t,ij} = \sigma_{t,ji}$):
  \[
  \Sigma_t = \begin{pmatrix}
  \sigma_{t,11} & \sigma_{t,12} & \sigma_{t,13} \\
  \sigma_{t,12} & \sigma_{t,22} & \sigma_{t,23} \\
  \sigma_{t,13} & \sigma_{t,23} & \sigma_{t,33}
  \end{pmatrix}
  \]
Portfolios

- If $w$ is the vector of portfolio weights
- The portfolio variance is

$$\sigma_{\text{portfolio}}^2 = w' \Sigma w$$
The curse of dimensionality

• Number of diagonal elements is $K$ and off diagonal elements $K(K - 1)/2$ so in all

$$K + K(K - 1)/2$$

• For two assets it is $2+1$, for 3 assets $3+4$, for 4 $10$, etc.
• The explosion in the number of variance and especially covariance terms, as the number of assets increases, is known as the *curse of dimensionality*
• This is one reason why it is more difficult to estimate the covariance matrix
Positive semi-definiteness

• For univariate volatility we need to ensure that the variance is not negative ($\sigma^2 \geq 0$)

• And for a portfolio

$$\sigma_{portfolio}^2 = w'\Sigma w \geq 0$$

• So a covariance matrix should be positive semi-definite:

$$|\Sigma| \geq 0$$

• This can be difficult to ensure
What about multivariate (MV) GARCH?

- For one asset
  \[ \sigma^2_{t+1} = \omega + \alpha y^2_t + \beta \sigma^2_t \]

- For two
  \[
  \begin{align*}
  \sigma^2_{t+1,1} &= \omega_1 + \alpha_1 y^2_{t,1} + \beta_2 \sigma^2_{t,1} + \alpha_2 y^2_{t,2} + \beta_2 \sigma^2_{t,2} + \delta_1 \sigma^2_{t+1,1,2} + \gamma_1 y_{t,1} y_{t,2} \\
  \sigma^2_{t+1,2} &= \omega_2 + \alpha_3 y^2_{t,1} + \beta_3 \sigma^2_{t,1} + \alpha_4 y^2_{t,2} + \beta_4 \sigma^2_{t,2} + \delta_2 \sigma^2_{t+1,1,2} + \gamma_2 y_{t,1} y_{t,2} \\
  \sigma^2_{t+1,1,2} &= \omega_3 + \alpha_5 y^2_{t,1} + \beta_5 \sigma^2_{t,1} + \alpha_6 y^2_{t,2} + \beta_6 \sigma^2_{t,2} + \delta_3 \sigma^2_{t+1,1,2} + \gamma_3 y_{t,1} y_{t,2}
  \end{align*}
\]

- Or 21 parameters to estimate
- Almost impossible in practice
Numerical issues

- Stationarity is more important for multivariate volatility.
- For univariate GARCH model, violation of covariance stationarity does not hinder the estimation process with numerical problems.
- A univariate volatility forecast is still obtained even if $\alpha + \beta > 1$.
- This is generally not the case for MV GARCH models.
- A parameter set resulting in violation of covariance stationarity might also lead to unpleasant numerical problems.
- Numerical algorithms need to address these problems, thus complicating the programming process considerably.
- Problems with multiple local minima, flat surfaces and other pathologies discussed in the last chapter.
Instead

- Use some simplification approaches
- Unfortunately they come with significant trade-offs
- So MV estimation is much harder and much less accurate than univariate estimation

1. EWMA
2. CCC
3. DCC (perhaps most widely used for portfolios)
4. OGARCH (perhaps most widely used for combining portfolios)
5. BEKK (not to be recommended except for very small problems, $K = 2$, perhaps $K = 3$)
6. MV GARCH (practically impossible except maybe when $K = 2$)
EWMA
**EWMA model**

- **Univariate:**
  \[ \hat{\sigma}_t^2 = \lambda \hat{\sigma}_{t-1}^2 + (1 - \lambda)y_{t-1}^2 \]

- A vector of returns is
  \[ \mathbf{y}_t = [y_{t,1}, y_{t,2}, \ldots, y_{t,K}] \]

- The multivariate EWMA is:
  \[ \hat{\Sigma}_t = \lambda \hat{\Sigma}_{t-1} + (1 - \lambda)\mathbf{y}_{t-1}'\mathbf{y}_{t-1} \]
  with an individual element given by:
  \[ \hat{\sigma}_{t,ij} = \lambda \hat{\sigma}_{t-1,ij} + (1 - \lambda)y_{t-1,i}y_{t-1,j} \]
Properties

- The same weight, $\lambda$, is used for all assets
- It is pre-specified and not estimated
- The variance of any particular asset only depends on its own lags
Pros and cons of multivariate EWMA model

Usefulness:

• Straightforward implementation, even for a large number of assets
• Covariance matrix is guaranteed to be positive semi-definite
Pros and cons of multivariate EWMA model

Usefulness:

- Straightforward implementation, even for a large number of assets
- Covariance matrix is guaranteed to be positive semi-definite

Drawbacks:

- Restrictiveness:
  - Simple structure
  - The assumption of a single and usually non-estimated $\lambda$
Multivariate volatility

EWMA = matrix(nrow=dim(y)[1], ncol=3)

lambda = 0.94

S = cov(y)

EWMA[1,] = c(S)[c(1,4,2)]

for (i in 2:dim(y)[1]){
    S = lambda*S+(1-lambda)*y[i-1,] %*% t(y[i-1,])
    EWMA[i,] = c(S)[c(1,4,2)]
}

EWMARho = EWMA[,3]/sqrt(EWMA[,1]*EWMA[,2])

Be careful with the matrix multiplications, getting the transposes right (y[i-1,] %*% t(y[i-1,])) and not t(y[i-1]) %*% y[i-1]
Constant Conditional Correlation Models
Steps

- De-mean returns
- Separate out correlation modelling from volatility modelling
  1. correlation matrix
  2. variances
- Model volatilities with GARCH or some standard method
- The correlation matrix can be static (CCC) or dynamic (DCC)
Definitions

• Let $D_t$ be a diagonal matrix where each element is the \textit{volatility} of each asset

$$D_{t,ii} = \sigma_{t,i}, \quad i = 1, \ldots, K$$
$$D_{t,ij} = 0, \quad i \neq j$$

• Use univariate GARCH (or some method) to estimate the variance of each asset separately, i.e. to get $D_t$

$$D_{t,ii} = \sigma_{t,i} = \sqrt{\omega_i + \alpha_i y_{t-1,i}^2 + \beta_i \sigma_{t-1,i}^2}, \quad i = 1, \ldots, K$$
• We want the correlations of the residuals

\[ \epsilon_{t,i} := \frac{y_{t,i}}{D_{t,i}} \]

\[ \epsilon_t \overset{K\times 1}{=} D_t^{-1}y_t \]

• And

\[ \epsilon \overset{T\times K}{=} \]

• Then the correlations are

\[ \hat{R} := \text{Cov}(\epsilon) = \frac{\epsilon\epsilon_t'}{T} \]
Constant conditional correlations (CCC)

- Then combine these two

\[ \hat{\Sigma}_t = \hat{D}_t \hat{R} \hat{D}_t \]

- Note that while \( D_t \) is time dependent, \( R \) is not
Pros and cons

**Pros**
- Guarantees the positive definiteness of $\hat{\Sigma}_t$ if $\hat{R}$ is positive definite
- Simple model, easy to implement
- Since matrix $\hat{D}_t$ has only diagonal elements, we can estimate each volatility separately

**Cons**
- The assumption of correlations being constant over time is at odds with the vast amount of empirical evidence supporting nonlinear dependence
Dynamic Conditional Correlation Models
Dynamic conditional correlations (DCC)

- DCC model is an extension of the CCC
- Let the correlation matrix $\hat{R}_t$ be _time dependent_
- While one might propose a model like

$$R_t = a + b\epsilon_{t-1}\epsilon'_{t-1} + cR_{t-1}$$

- That will not work since we have to ensure $|\Sigma_t| > 0$, i.e. positive definite. For that:
  - $D_t$ is positive definite by construction since all elements in $D_t$ are positive
  - All elements in $R_t$ need to be $\leq 1$ and $\geq -1$
So need more steps

- Decompose $R_t$ into

$$R_t = Q_t^* Q_t Q_t^*$$

- $Q_t$ is a positive definite matrix that dives the dynamics
- $Q_t^*$ re-scales $Q_t$ to ensure each element $|q_{t,ij}| < 1$

$$Q_t^* = \begin{pmatrix}
1/\sqrt{q_{t,11}} & 0 & \cdots & 0 \\
0 & 1/\sqrt{q_{t,22}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1/\sqrt{q_{t,\text{KK}}} \\
\end{pmatrix}$$
The have $Q_t$ follow an ARMA type process

$$Q_t = (1 - \zeta - \xi)\overline{Q} + \zeta \epsilon_{t-1} \epsilon'_{t-1} + \xi Q_{t-1}$$

- $\overline{Q}$ is the $(K \times K)$ unconditional covariance matrix of $\epsilon$
- $\zeta$ and $\xi$ are parameters
- Parameter restrictions:
  - Positive definiteness $\zeta, \xi > 0$
  - Stationarity $\zeta + \xi < 1$
Pros and cons of DCC

Pros

• Large covariance matrices can be easily be estimated
Pros and cons of DCC

Pros
• Large covariance matrices can be easily estimated

Cons
• Parameters $\zeta$ and $\xi$ are constants
• So the conditional correlations of all assets are driven by the same underlying dynamics, parameters are same for all assets
R estimation

```
library(rmgarch)

xsdoc = ugarchspec(
  mean.model = list(
    armaOrder = c(0, 0),
    include.mean = FALSE)
)

uspec = multispec(replicate(2, xsdoc))

spec = dccspec(uspec = uspec,
  dccOrder = c(1, 1),
  distribution = 'mvnorm')

res = dccfit(spec, data = y)
```
Getting correlations

- \texttt{res@mfit}$H$ contains the covariance matrices
- It is a $2 \times 2 \times T$ dimensional matrix

\begin{verbatim}
H = \texttt{res@mfit}$H$

rho = \texttt{vector}(\textit{length}=\texttt{dim}(y)[1])

for (i in 1:dim(y)[1]) {
    rho[i] = H[1,2,i] / \texttt{sqrt}(H[1,1,i] \times H[2,2,i])
}
\end{verbatim}
Large problems
Large problems

• Even a medium-sized financial institutions will have hundreds of thousands or millions of types of assets — what is known as risk factors
• Estimating the covariance matrix for the entire institution is effectively impossible using methods like EWMA or DCC
• Instead, we can split the risk factors up into subcomponents
• Estimate the covariance of each
• And then combine them back
For example

- US
  - Bonds
  - Commodity
  - S&P 100 Index
  - Equities
    - S&P 400 Mid Cap Index

- Bank
  - China
    - Bonds
    - Equities
  - Brazil
    - Bonds
    - Equities
• The orthogonal approach transforms the observed returns matrix into a set of portfolios with the key property that they are *uncorrelated*

• We can forecast their volatilities separately

• Principal components analysis (PCA)

• Known as orthogonal GARCH, or OGARCH

• Because it involves transforming correlated returns into uncorrelated portfolios and then using GARCH to forecast the volatilities of each uncorrelated portfolio separately
Idea

• We have a matrix of returns

\[ Y_{T \times K} \]

• With covariance matrix

\[ \Sigma_{K \times K} \]

• Let

\[ D = \sqrt{\text{diag} \Sigma} \]

• Then the correlations are

\[ R = D^{-1} \Sigma D^{-1} \]
Orthogonalizing covariance

Making covariance uncorrelated

- Transform the return matrix $\mathbf{Y}$ into uncorrelated portfolios $\mathbf{U}$
- Calculate the $K \times K$ matrix of eigenvectors of $\mathbf{R}$
- Denote the matrix $\Lambda$
- $\mathbf{U}$ is defined as:

$$\mathbf{U} = \Lambda \times \mathbf{Y}$$
Large-scale implementations

- Large number of assets
- The method also allows estimates for volatilities and correlations of variables to be generated even when data are sparse (e.g., in illiquid markets)
- The use of PCA guarantees the positive definiteness of the covariance matrix
- PCA also facilitates building a covariance matrix for an entire financial institution by iteratively combining the covariance matrices of the various trading desks, simply by using one or perhaps two PCs
Consider an example...

• Your financial institution has the following securities:
Consider an example...

- Your financial institution has the following securities:

\[ \sigma^2_E \]
Consider an example...

- Your financial institution has the following securities:

\[ \sigma_E^2 \quad \sigma_B^2 \]
Consider an example...

- Your financial institution has the following securities:

\[ \sigma^2_E, \sigma^2_B, \sigma^2_E \]
Consider an example...

• Your financial institution has the following securities:

\[ \sigma_E^2, \sigma_B^2, \sigma_E^2, \sigma_B^2 \]
Consider an example...

- Your financial institution has the following securities:

\[
\begin{align*}
\sigma^2_E & \quad \sigma^2_B \\
\sigma^2_E & \quad \sigma^2_B
\end{align*}
\]

where red is UK, blue is Germany, E denotes EQUITY and B is BOND
Consider an example...

- Your financial institution has the following securities:
  \[ \sigma_E^2 \quad \sigma_B^2 \quad \sigma_E^2 \quad \sigma_B^2 \]
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- Individual covariance matrices of your securities can be combined into a covariance matrix for the entire financial institution:
Consider an example...

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\begin{align*}
\sigma_E^2 & \\
\sigma_B^2 & 
\end{align*}
\]
Consider an example...

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where red is UK, blue is Germany, E denotes EQUITY and B is BOND

• Individual covariance matrices of your securities can be combined into a covariance matrix for the entire financial institution:

\[ \sigma_E^2 \quad \rho_{E,B} \quad \sigma_B^2 \]
Consider an example...

- Your financial institution has the following securities:

\[
\begin{array}{cccc}
\sigma^2_E & \sigma^2_B & \sigma^2_E & \sigma^2_B \\
\end{array}
\]

where red is UK, blue is Germany, E denotes EQUITY and B is BOND

- Individual covariance matrices of your securities can be combined into a covariance matrix for the entire financial institution:

\[
\begin{array}{ccc}
\sigma^2_E & \rho_{E,B} & \sigma^2_B \\
\end{array}
\]

\[
\begin{array}{c}
\sigma^2_E \\
\end{array}
\]
Consider an example...

- Your financial institution has the following securities:

\[
\begin{bmatrix}
\sigma_E^2 & \sigma_B^2 \\
\sigma_E^2 & \sigma_B^2 \\
\end{bmatrix}
\]

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\[
\begin{bmatrix}
\sigma_E^2 & \rho_{E,B} & \rho_{E,E} & \rho_{B,E} \\
\rho_{E,B} & \sigma_B^2 \\
\rho_{E,E} & \rho_{B,E} & \sigma_E^2 \\
\end{bmatrix}
\]
Consider an example...

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\[
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\[
\begin{array}{cccc}
\sigma_E^2 & & & \\
\rho_{E,B} & \sigma_B^2 & & \\
\rho_{E,E} & \rho_{B,E} & \sigma_E^2 & \\
& & \sigma_B^2 & \\
\end{array}
\]
Consider an example...

- Your financial institution has the following securities:

\[
\begin{array}{ccc}
\sigma_E^2 & \sigma_B^2 & \sigma_E^2 \\
\sigma_B^2 & \rho_{E,B} & \sigma_B^2 \\
\rho_{E,E} & \rho_{B,E} & \sigma_E^2 \\
\rho_{B,B} & \rho_{E,B} & \sigma_B^2 \\
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\rho_{B,B} & \rho_{E,B} & \sigma_B^2 \\
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  \rho_{E,B} & \sigma_B^2 & \rho_{E,E} & \rho_{B,E} \\
  \rho_{E,B} & \rho_{B,E} & \sigma_E^2 & \sigma_B^2 \\
  \rho_{B,B} & \rho_{E,B} & \sigma_B^2 & \sigma_E^2 \\
  \end{pmatrix}
  \]
  where red is UK, blue is Germany, E denotes EQUITY and B is BOND.

- Individual covariance matrices of your securities can be combined into a covariance matrix for the entire financial institution:
Consider an example...

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\rho_{E,E} & \rho_{B,E} & \rho_{B,B} & \rho_{E,B} \\
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- Individual covariance matrices of your securities can be combined into a covariance matrix for the entire financial institution:
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</table>
GO-GARCH

- GO-GARCH (van der Weide, 2002)
- Represent returns as factors

\[ Y = AF \]

- \( A \) is invertible and constant over time
- Rows of \( A \) are factor weights of each asset, rows are assets, columns factors
- \( \Sigma \) unconditional covariance

\[ A = \Sigma^{1/2} U \]

- The factors are then

\[ f_t = H_t^{1/2} z_t \]

- With \( H \) a diagonal matrix of asset variances \( h_{i,t} \)
- Returns are then

\[ y_t = AH_t^{1/2} \epsilon_t \]
R estimation

```r
library(rmgarch)
spec = gogarchspec(
  mean.model = list(armaOrder = c(0, 0), include.mean = FALSE),
  variance.model = list(model = "sGARCH", garchOrder = c(1,1)),
  distribution.model = "mvnorm"
)
fit = gogarchfit(spec = spec, data = y)
```
Estimation Comparison
Prices

Prices for MSFT and IBM from 2000 to 2015.
(b) Returns

MSFT

IBM


-15% -10% -5% 0% 5% 10% 15%


-15% -10% -5% 0% 5% 10% 15%
Correlation estimates

with average correlation 49%
Correlation estimates
with average correlation 49%

-20%
0%
20%
40%
60%
80%


EWMA
DCC

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• Let’s focus on 2008, the midst of the 2007-2009 crisis, when the correlations of all stocks increased dramatically...
Prices

- MSFT
- IBM
Returns

MSFT

IBM
Correlations

with average correlation 49%
Correlations

with average correlation 49%
BEKK
The BEKK model

• An alternative to the MV-GARCH models
• The matrix of conditional covariances is $\Sigma_t$
• A function of the outer product of lagged returns and lagged conditional covariances
• Each pre-multiplied and post-multiplied by a parameter matrix
• Results in a quadratic function that is guaranteed to be positive semi-definite
The two-asset, one-lag BEKK(1,1,2) model is defined as:

\[
\Sigma_t = \Omega \Omega' + A' Y_{t-1} Y_{t-1} A + B' \Sigma_{t-1} B
\]

or:

\[
\Sigma_t = \begin{pmatrix}
\sigma_{t,11} & \sigma_{t,12} \\
\sigma_{t,12} & \sigma_{t,22}
\end{pmatrix} = \begin{pmatrix}
\omega_{11} & 0 \\
\omega_{21} & \omega_{22}
\end{pmatrix}
\begin{pmatrix}
\omega_{11} & 0 \\
\omega_{21} & \omega_{22}
\end{pmatrix}'
\]

\[
+ \begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{pmatrix}' \begin{pmatrix}
Y^2_{t-1,1} & Y_{t-1,1} Y_{t-1,2} \\
Y_{t-1,2} Y_{t-1,1} & Y^2_{t-1,2}
\end{pmatrix}
\begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & \alpha_{22}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22}
\end{pmatrix}' \begin{pmatrix}
\sigma_{t-1,11} & \sigma_{t-1,12} \\
\sigma_{t-1,12} & \sigma_{t-1,22}
\end{pmatrix}
\begin{pmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22}
\end{pmatrix}
\]
• The general BEKK($L_1, L_2, K$) model is given by:

\[
\Sigma_t = \Omega \Omega' + \sum_{k=1}^{K} \sum_{i=1}^{L_1} A'_{i,k} Y'_{t-i} Y_{t-i} A_{i,k} + \sum_{k=1}^{K} \sum_{j=1}^{L_2} B'_{j,k} \Sigma_{t-j} B_{j,k}
\]

• The number of parameters in the BEKK(1,1,2) model is $K(5K + 1)/2$
  • 11 in two asset case
  • 24 in three asset case
  • 42 in four asset case
Pros and cons

Pros

• Allows for interactions between different asset returns and volatilities
• Relatively parsimonious
Pros and cons

**Pros**
- Allows for interactions between different asset returns and volatilities
- Relatively parsimonious

**Cons**
- Parameters hard to interpret
- Many parameters are often found to be statistically insignificant, which suggests the model may be overparametrized
- Can only handle a small number of assets
Matlab estimation

\[
[\text{PARAMETERS}, \text{LL, HT}] = \text{bekk}(y, [], 1, 0, 1);
\]

This took 33 seconds on my laptop