What is risk?

Value-at-Risk

Issues

Coherence

Manipulation

ES

Scaling

Regulations
The focus of this chapter is on

- Defining and measuring risk
  - Volatility
  - VaR (Value-at-Risk)
  - ES (Expected Shortfall)
- Holding periods
- Scaling and the square-root-of-time
Notation

\[ p \] Probability
\[ Q \] Profit and loss
\[ q \] Observed profit and loss
\[ w \] Vector of portfolio weights
\[ X \text{ and } Y \] Refer to two different assets
\[ \varphi(.) \] Risk measure
\[ \vartheta \] Portfolio value
Defining Risk
General Definition

- No universal definition of what constitutes risk
- On a very general level, financial risk could be defined as “the chance of losing a part or all of an investment”
- Large number of such statements could equally be made, many of which would be contradictory
Which asset do you prefer?

All three assets have volatility one and mean zero
Which asset do you prefer?

All three assets have volatility one and mean zero
Which asset do you prefer?

All three assets have volatility one and mean zero.
Which asset do you prefer?

- *Standard mean variance analysis* indicates that all three assets are equally risky and preferable
- Since we have the same mean
  \[
  E(A) = E(B) = E(C) = 0
  \]
- And the same volatility
  \[
  \sigma_A = \sigma_B = \sigma_C = 1
  \]
Which asset is “better”?  

- There is no obvious way to discriminate between the assets  
- One can try to model the *underlying distribution* of market prices and returns of assets, but it is generally *unknown*.  
  - can identify by maximum likelihood methods  
  - or test the distribution against other other distributions by using methods such as the Kolmogorov-Smirnov test  
- Practically, it is impossible to accurately identify the distribution of financial returns
Risk is a latent variable

- Financial risk is cannot be measured directly
- Risk has to be *inferred* from the behavior of observed market prices
  - e.g. at the end of a trading day, the return of the day is known while the risk is unknown
Risk measure and risk measurement

Risk measure  a mathematical concept of risk
Risk measurement  a number that captures risk, obtained by applying data to a risk measure
Volatility

- Volatility is the *standard deviation of returns*
- Main measure of risk in most financial analysis
- It is a *sufficient* measure of risk when returns are *normally distributed*
  - For this reason, in mean-variance analysis the efficient frontier shows the best investment decision
  - If returns are not normally distributed, solutions on the efficient frontier may be inefficient
Volatility

- The assumption of *normality of return is violated* for most if not all financial returns
  - See Chapter 1 on the non-normality of returns
- For most applications in financial risk, volatility is likely to systematically *underestimate risk*
Value–at–Risk (VaR)
**Definition: Value-at-Risk.** The loss on a trading portfolio such that there is a probability $p$ of losses equaling or exceeding VaR in a given trading period and a $(1 - p)$ probability of losses being lower than the VaR.

- The most common risk measure after volatility
- It is *distribution independent*
Quantiles and P/L

- VaR is a \textit{quantile} on the distribution of P/L (profit and loss)
- We indicate the P/L on an investment portfolio by the random variable $Q$, with a realization indicated by $q$
- In the case of holding \textit{one unit} of an asset, we have

$$Q_t = P_t - P_{t-1}$$

- More generally, if the portfolio value is $\vartheta$:

$$Q_t = \vartheta Y_t = \vartheta \frac{P_t - P_{t-1}}{P_{t-1}}$$

- That is, the P/L is the portfolio value ($\vartheta$) multiplied by the returns
VaR and P/L density

- The density of P/L is denoted by $f_q(.)$, then VaR is given by:

$$\Pr[Q \leq -\text{VaR}(p)] = p$$

or,

$$p = \int_{-\infty}^{-\text{VaR}(p)} f_q(x) \, dx$$

- We usually write it as VaR($p$) or $\text{VaR}^{100\times p\%}$
  - for example, VaR(0.05) or VaR$^5\%$
Is VaR a negative or positive number?

- VaR can be stated as a negative or positive number.
- Equivalently, probabilities can be stated as close to one or close to zero – for example, VaR(0.95) or VaR(0.05).
- We take the more common approach of referring to VaR as a *positive number* using *low-probability terminology* (e.g. 5%).
VaR graphically

Density of profit and loss
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VaR graphically

Density zoomed in

$-\text{VaR} 1\%$

$-\text{VaR} 5\%$

$p=1\%$

$p=5\%$
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VaR graphically

Cumulative distribution of profit and loss
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**VaR graphically**

Cumulative distribution zoomed in

- **-VaR 1%**
- **-VaR 5%**
Example

- The commodities’ trading book is worth £1 billion and daily $\text{VaR}^{1\%} = £10$ million
- This means we expect to lose £10 million or more once every 100 days, or about once every 5 months
The three steps in VaR calculations

1. The *probability* of losses exceeding VaR, $p$
2. The *holding period*, the time period over which losses may occur
3. The *probability distribution* of the P/L of the portfolio
Which probability should we use?

- VaR levels of 1% – 5% are very common in practice
- Regulators (Basel II) demand 1%
- But less extreme numbers, such as 10% are often used in risk management on the trading floor
- More extreme lower numbers, such as 0.1%, may be used for applications like economic capital, survival analysis or long-run risk analysis for pension funds
Holding period

- The *holding period* is the time period over which losses may occur
  - it is usually one day
  - can be minutes or hours
  - or several days, but it does not make sense to use more than 2 weeks, and even that is on the high side

- Holding periods can vary depending on different circumstances

- Many proprietary trading desks focus on intraday VaR

- For institutional investors and nonfinancial corporations, it is more realistic to use longer holding periods
Probability distribution of P/L

- The identification of the probability distribution is difficult
- The standard practice is to estimate the distribution by using past observations and a statistical model
- We will use EWMA and GARCH later
VaR and normality

- VaR *does not* implies normality of returns, we can use any distribution in calculating VaR
- Even if I am often surprised by people who assume it does
- However, the most common distribution assumption for returns in the calculation of VaR is *conditional* normality
  - In this case, volatility provides the same information as VaR
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Sign of VaR

- If the mean of the density of P/L is sufficiently large, the probability \( p \) quantile (the VaR), might easily end up on the other side of zero.
- This means that the relevant losses have become profits.
- In such situations we either specify a more extreme \( p \) or use a different measure of risk.
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Sign of VaR

\[ f(q) \]

\[ -\text{VaR}^{5\%} \quad \mu = 0 \]
Sign of VaR

\[ f(q) \]

\[ f(q) \]

\[ -\text{VaR}^{5\%} \quad \mu = 0 \]

suppose mean is bigger
Sign of VaR

Suppose mean is bigger

\[ f(q) \]

\[ \mu = 4 \]
Sign of VaR

Note that -VaR is now positive
Issues in applying VaR
Main issues in the implementation

1. VaR is only a quantile on the P/L distribution
2. VaR is not a coherent risk measure
3. VaR is easy to manipulate
VaR is only a quantile

- VaR gives the "best of worst case scenarios" and, as such, it inevitably underestimates the potential losses associated with a probability level.
- I.e. VaR($p$) is incapable of capturing the risk of extreme movements that have a probability of less than $p$.
- If VaR=$1000$, are potential losses $1001$ or $10000000$?
- The shape of the tail before and after VaR need not have any bearing on the actual VaR number.
VaR in unusual cases

\[ f(q) \]

\[ -\text{VaR}^{5\%} \]
VaR in unusual cases
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VaR in unusual cases

\[ f(q) \]

\[ -\text{VaR}^{5\%} \quad 0 \quad q \]

\[ f(q) \]

\[ -\text{VaR}^{5\%} \quad 0 \quad q \]

\[ f(q) \]

\[ -\text{VaR}^{5\%} \quad 0 \quad q \]
VaR in unusual cases

\[ f(q) \]

- \( \text{VaR}^{5\%} \)

\[ f(q) \]

- \( \text{VaR}^{5\%} \)
The ideal properties of any financial risk measure were proposed by Artzner et. al (1999)

To them, coherence is ideal

Other authors have added more “ideal conditions”

While others have dismissed the importance of some of these
Coherence

• Suppose we have two assets, $X$ and $Y$
• Denote some arbitrary risk measure by $\varphi(\cdot)$. It could be volatility, VaR or something else
• $\varphi(\cdot)$ is then some function that maps some observations of an asset, like $X$, onto a risk measurement
• Further define some arbitrary constant $c$
• We say that $\varphi(\cdot)$ is a coherent risk measure if it satisfies the following four axioms
  1. Monotonicity
  2. Translation invariance
  3. Positive homogeneity
  4. Subadditivity
Monotonicity

- If
  \[ X \leq Y \]
- and
  \[ \varphi(X) \geq \varphi(Y) \]
- Then risk measure \( \varphi \) satisfies monotonicity
- What this means is that if outcomes for asset \( X \) are always more negative than outcomes for \( Y \)
  - suppose \( X \) and \( Y \) are daily returns on AMZN and GOOG, and the returns on GOOG are always higher than for AMZN, then GOOG risk is lower than that of AMZN
  - Then the risk of \( Y \) should never exceed the risk of \( X \)
  - This is perfectly reasonable and should always hold
Translation invariance

- If
  \[ \varphi(X + c) = \varphi(X) - c \]
- Then risk measure \( \varphi \) satisfies translation invariance
- In other words, if we add a positive constant to the returns of AMZN then the risk will go down by that constant
- This is perfectly reasonable and should always hold
Positive homogeneity

- If \( c > 0 \) and
  \[ \varphi(cX) = c\varphi(X) \]
- Then risk measure \( \varphi \) satisfies positive homogeneity
- Positive homogeneity means risk is *directly proportional* to the value of the portfolio
- For example, suppose a portfolio is worth $1,000 with risk $10, then doubling the portfolio size to $2,000 will double the risk to $20
As relative shareholdings increase, the risk may increase more rapidly than the portfolio size. In this case, *positive homogeneity is violated*:

$$\varphi(cX) > c\varphi(X)$$

This is because when we are trying to sell, the price of the stock falls, therefore the eventual selling price is lower than the initial market price

- See chapter 10, Endogenous risk
Subadditivity

- If
  \[ \phi(X + Y) \leq \phi(X) + \phi(Y) \]
- Then risk measure \( \phi \) satisfies Subadditivity
- Subadditivity means a portfolio of assets is measured as less risky than the sum of the risks of individual assets
- That is, diversification reduces risk
Volatility is subadditive

- Recall how portfolio variance is calculated when we have two assets
- \( X \) and \( Y \), with volatilities \( \sigma_X \) and \( \sigma_Y \), respectively, correlation coefficient \( \rho \) and portfolio weights \( w_X \) and \( w_Y \)
The portfolio variance is:

\[ \sigma_{\text{portfolio}}^2 = w_X^2 \sigma_X^2 + w_Y^2 \sigma_Y^2 + 2w_X w_Y \rho \sigma_X \sigma_Y \]

Rewriting, we get

\[ \sigma_{\text{portfolio}}^2 = (w_X \sigma_X + w_Y \sigma_Y)^2 - 2w_X w_Y (1 - \rho) \sigma_X \sigma_Y \]

where the last term is positive

- \( W_X, W_Y \geq 0, -1 \leq \rho \leq 1 \)
- Volatility is therefore \textit{subadditive} because:

\[ \sigma_{\text{portfolio}} \leq w_X \sigma_X + w_Y \sigma_Y \]
VaR can violate subadditivity

- Asset $X$ has probability of 4.9% of a return of -100, and 95.1% of a return of 0
- Hence we have

$$\text{VaR}^{5\%}(X) = 0$$
$$\text{VaR}^{1\%}(X) = 100$$
Consider another asset $Y$, independent of $X$ and with the same distribution as $X$.

Suppose we hold an equally weighted portfolio of assets $X$ and $Y$, the 5% VaR of the portfolio is

$$\text{VaR}^{5\%}_{\text{portfolio}} = \text{VaR}^{5\%}(0.5X + 0.5Y) = 50$$

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$X$</th>
<th>$Y$</th>
<th>$\frac{1}{2}X + \frac{1}{2}Y$</th>
<th>Probability</th>
<th>Cumulative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-100</td>
<td>-100</td>
<td>-100</td>
<td>0.2%</td>
<td>0.2%</td>
</tr>
<tr>
<td>2</td>
<td>-100</td>
<td>0</td>
<td>-50</td>
<td>4.7%</td>
<td>4.9%</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-100</td>
<td>-50</td>
<td>4.7%</td>
<td>9.6%</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>90.4%</td>
<td>100%</td>
</tr>
</tbody>
</table>
• In this case, VaR\(^{5\%}\) violates subadditivity

\[ \text{VaR}_{\text{portfolio}}^{5\%} > 0.5 \text{VaR}^{5\%}(X) + 0.5 \text{VaR}^{5\%}(Y) = 0 \]

• This is because the probability of a loss (4.9%) of a single asset is slightly smaller than the VaR probability (5%)
• While the portfolio probability is higher than 5%
Does VaR really violate subadditivity?

- VaR is subadditive in the special case of normally distributed returns
- Subadditivity for the VaR is violated when the tails are super fat
  - For example a Student-t where the degrees of freedom are less than one
  - Imagine you go to a buffet restaurant where you suspect one of the dishes might give you food poisoning
  - Then the optimal strategy is only to eat one dish, not to diversify
- Most assets do not have super fat tails, this includes most equities, exchange rates and commodities
• VaR of assets that are subject to occasional very large negative returns tends to suffer subadditivity violations, e.g.
  • Exchange rates in countries that peg their currency but are subject to occasional devaluations
  • Electricity prices subject to very extreme price swings
  • Junk bonds where most of the time the bonds deliver a steady positive return
  • Short deep out of the money options

• Enough for this to apply to one asset in a portfolio
Manipulating VaR

- VaR is easily it can be manipulated, perhaps to make the VaR measurement lower without risk it self falling
- There are many ways to do this, for example
  1. cherry pick assets that make a VaR measure low
  2. particular derivative trading strategies
VaR manipulation with derivatives

- Suppose the VaR before any manipulation is $\text{VaR}_0$ and that a bank prefers the VaR to be $\text{VaR}_1$
- Where, $0 < \text{VaR}_1 < \text{VaR}_0$
VaR manipulation with derivatives

- Suppose the VaR before any manipulation is $\text{VaR}_0$ and that a bank prefers the VaR to be $\text{VaR}_1$
- Where, $0 < \text{VaR}_1 < \text{VaR}_0$
• This can be achieved by
  1. buying put with a strike above $\text{VaR}_1$
  2. writing a put option with a strike price below $\text{VaR}_0$
• This will result a lower expected profit
  • the fee from writing the option is lower than the fee from buying the option
• And an increase in downside risk
  • because it the potential for large losses (makes the tail fatter)
• While this may be an obvious manipulation
• It can be very hard to identify in the real world
Expected Shortfall (ES)
Expected Shortfall (ES)

- A large number of other risk measures have been proposed
- The only one to get traction is ES
- It is known by several names, including
  1. ES
  2. Expected tail loss
  3. Tail VaR
**Definition: Expected shortfall.** Expected loss conditional on VaR being violated (i.e. expected P/L, $Q$, when it is lower than negative VaR)

$$ES = -E[Q|Q \leq -\text{VaR}(p)]$$

- ES is an alternative risk measures to VaR which overcomes the problem of subadditivity violation
- It is aware of the shape of the tail distribution while VaR is not
ES and VaR for profit/loss outcomes

Density of P/L and VaR

the shaded area is 0.05
ES and VaR for profit/loss outcomes

Left tail of the density

Take expectation over the shaded area
ES and VaR for profit/loss outcomes

Blow up the tail

Scale tail area up to 1
Expected Shortfall

- **ES**
  \[ \text{ES} = \int_{-\infty}^{-\text{VaR}(p)} x f_{\text{VaR}}(x) \, dx \]

- The tail density, \( f_{\text{VaR}}(.) \) is given by:
  \[ 1 = \int_{-\infty}^{-\text{VaR}(p)} f_{\text{VaR}}(x) \, dx = \frac{1}{p} \int_{-\infty}^{-\text{VaR}(p)} f_q(x) \, dx \]
Under the standard normal distribution

- If the portfolio value is 1, then:

\[
ES = -\frac{\phi(\Phi^{-1}(p))}{p}
\]

where \(\phi\) and \(\Phi\) are the normal density and distribution, respectively.

- VaR and ES for different levels of confidence for a portfolio with a face value of $1 and normally distributed P/L with mean 0 and volatility 1:

<table>
<thead>
<tr>
<th>(p)</th>
<th>0.5</th>
<th>0.1</th>
<th>0.05</th>
<th>0.025</th>
<th>0.01</th>
<th>0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>VaR</td>
<td>0</td>
<td>1.282</td>
<td>1.645</td>
<td>1.960</td>
<td>2.326</td>
<td>3.090</td>
</tr>
<tr>
<td>ES</td>
<td>0.798</td>
<td>1.755</td>
<td>2.063</td>
<td>2.338</td>
<td>2.665</td>
<td>3.367</td>
</tr>
</tbody>
</table>
Pros and cons

Pros

- ES in *coherent* and VaR is not
- It is *harder to manipulate* ES than VaR

Cons

- ES is measured with *more uncertainty* than VaR
- ES is much *harder to backtest* than VaR because the ES procedure requires estimates of the tail expectation to compare with the ES forecast
Holding Periods, Scaling and the Square–Root–of–Time
Length of holding periods

- In practice, *the most common holding period is daily*
- Shorter holding periods are common for risk management on the trading floor
  - where risk managers use hourly, 20-minute and even 10 minute holding periods
  - this is technically difficult because intraday data has complicated diurnal patterns
Longer holding periods

- Holding periods exceeding one day are also demanding
  - the effective date the sample becomes much smaller
  - one could use scaling laws
- Most VaR forecasts require at least a few hundred observations to estimate risk accurately
- For a 10-day holding period will need at least 3,000 trading days, or about 12 years
- In most cases data from 12 years ago are fairly useless
Scaling laws

• If data comes from a particular stochastic process it may be possible to use VaR estimates at high frequency (e.g. daily) and scale them up to lower frequencies (e.g. biweekly)
• This would be possible because we know the stochastic process and how it aggregates
• That is not usually the case
Variances, IID and square–root–of–time scaling

- Suppose we observe an IID random variable \( \{X_t\} \) with variance \( \sigma^2 \) over time.
  - The variance of the sum of two consecutive \( X \)’s is then:
    \[
    \text{Var}(X_t + X_{t+1}) = \text{Var}(X_t) + \text{Var}(X_{t+1}) = 2\sigma^2
    \]
  - The scaling law for variances is time.
  - It is the same for the mean.
  - While the scaling law for standard deviations (volatility) is square root of time.
  - This holds regardless of the underlying distribution (provided the variance is defined).
Square-root-of-time scaling

- The square-root-of-time rule applies to volatility when data is IID
- It does not apply when data is not IID, like GARCH
- This rule applies to volatility regardless of the underlying distribution provided that the returns are IID
- For VaR, the square-root-of-time rule only applies returns are IID *normally* distributed
  - note we need an additional assumption
- It is possible to derive the scaling law for IID fat tailed data
Regulations
Regulations

- All financial institutions are regulated
- Banks are regulated under the Basel Accords
- Determined by the Basel Committee (Under G20)
- Countries commit themselves to implementing the Basel Accords
Basel Accords

- Three main risk factors
  1. trading book
  2. banking book
  3. operational risk
- Our focus here is on trading book
Trading book

- Basel II (in effect from about 2008 until 2019)
  - banks are required to measure market risk with VaR^{99%}
    with ten-day holding periods
- Basel III (in effect from about 2019 (some parts implemented earlier))
  - banks are required to measure market risk with ES^{97.5%}
    with various holding periods
Use of regulatory risk measures

- Ensure banks have effective risk management systems
- Identify if they are taking too much risk
- Financial institutions regulated under the Basel II Accords are required to set aside a certain amount of capital due to *market risk*, *credit risk* and *operational risk*
- It is based on multiplying the maximum of previous day 1% VaR and 60 days average VaR by a constant, $\Xi$, which is determined by the number of violations that happened previously:

$$\text{Market risk capital}_t = \Xi_t \max \left( \text{VaR}^{1\%}_t, \overline{\text{VaR}}^{1\%}_t \right) + \text{constant}$$
- VaR$_t^{1\%}$ is average reported 1% VaR over the previous 60 trading days
- The multiplication factor $\Xi_t$ varies with the number of violations, $\nu_1$, that occurred in the previous 250 trading days - the required testing window length for backtesting in the Basel Accords.
- This is based on three ranges for the number of violations, named after the three colors of traffic lights:

$$\Xi_t = \begin{cases} 
3, & \text{if } \nu_1 \leq 4 \text{ (Green)} \\
3 + 0.2(\nu_1 - 4), & \text{if } 5 \leq \nu_1 \leq 9 \text{ (Amber)} \\
4, & \text{if } 10 \leq \nu_1 \text{ (Red)}
\end{cases}$$