Financial Risk Forecasting
Chapter 6
Analytical value-at-risk for options and bonds

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The focus of this chapter

- Calculate VaR for options and bonds
  - Not possible with methods from Chapters 4 and 5
- We start by using analytical methods, deriving VaR mathematically
- Monte Carlo methods are discussed in Chapter 7
  - Preferred for most applications
**VaR for options and bonds**

- Chapters 4 and 5 showed how a VaR can be obtained an asset distribution.
- That is not possible for assets such as bonds and options, as their intrinsic value changes with passing of time.
  - e.g. the price of bond converges to fixed value as time to maturity elapses, so inherent risk decreases over time.
- Value of bonds and options is non–linearly related to the underlying asset.
The first two sections of these slides introduce the problem of the nonlinear relationship between the underlying asset and a bond and option.

The last two sections show how one can use mathematical approximations to obtain a closed form solution.

Generally, such methods are not recommended.

And is better to use the simulation methods in the next chapter.
Notation

\[ T \quad \text{Delivery time/maturity} \]
\[ r \quad \text{Annual interest rate} \]
\[ \sigma_r \quad \text{Volatility of daily interest rate increments} \]
\[ \sigma_a \quad \text{Annual volatility of an underlying asset} \]
\[ \sigma_d \quad \text{Daily volatility of an underlying asset} \]
\[ \tau \quad \text{Cash flow} \]
\[ D^* \quad \text{Modified duration} \]
\[ C \quad \text{Convexity} \]
\[ \Delta \quad \text{Option delta} \]
\[ \Gamma \quad \text{Option gamma} \]
\[ g(\cdot) \quad \text{Generic function name for pricing equation} \]
\[ \vartheta \quad \text{Portfolio value} \]
Bonds
Bond pricing

- A bond is a fixed income instrument
- Typically with regular payments
- Bond price is given by present value of future cash flows

\[
\sum_{t=1}^{T} \frac{\tau_t}{(1 + r_t)^t}
\]

- Where \(\{\tau_t\}_{t=1}^{T}\) includes the coupon and principal payments
- And \(r_t\) is the interest rate in each period
Bond risk asymmetry

- Bond has face value $1000, maturity of 50 years and annual coupon of $30
- Yield curve is flat, annual interest rates at 3%
- So its current price is equal to the par value
- Now consider parallel shifts in the yield curve to 1% or 5%

<table>
<thead>
<tr>
<th>Interest rate</th>
<th>Price</th>
<th>Change in price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>$1784</td>
<td>$784</td>
</tr>
<tr>
<td>3%</td>
<td>$1000</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>$635</td>
<td>-$365</td>
</tr>
</tbody>
</table>
Bond risk asymmetry

Interest rate

Bonds

Options

Duration VaR

Delta VaR

Summary

Introduction

1.0% 3.0% 5.0% 7.0%

$400 $600 $800 $1000 $1200 $1400 $1600 $1800

$1000

$1784

$635
Bond risk

- Change from 3% to 1% makes bond price increase by $784
- Change from 3% to 5% makes it fall by $365
Options
Options

• An option gives its owner the right, but not the obligation, to call (buy) or put (sell) an underlying asset at a strike price on a fixed expiry date
  • European options can only be exercised at expiration
  • American options can be exercised at any point up to expiration
• We will focus on European options, but the basic analysis could be extended to many other variants
Black-Scholes equation

Pricing European options

- Black and Scholes (1973) developed an equation for pricing European options
- Refer to the Black-Scholes (BS) pricing function as $g(\cdot)$
- We use the following notation:
  - $P_t$: Price of underlying asset at year $t$
  - $X$: Strike price
  - $r$: Annual risk-free interest rate
  - $T - t$: Time until expiration
  - $\sigma_a$: Annual volatility
  - $\Phi$: Standard normal distribution
• The BS function for an European option

\begin{align*}
\text{put}_t &= X e^{-r(T-t)} - P_t + \text{call}_t \\
\text{call}_t &= P_t \Phi (d_1) - X e^{-r(T-t)} \Phi (d_2)
\end{align*}

where

\begin{align*}
d_1 &= \frac{\log (P_t/X) + (r + \sigma^2_a/2) (T - t)}{\sigma_a \sqrt{T - t}} \\
d_2 &= \frac{\log (P_t/X) + (r - \sigma^2_a/2) (T - t)}{\sigma_a \sqrt{T - t}} \\
&= d_1 - \sigma_a \sqrt{T - t}
\end{align*}
• Value of an option is affected by many underlying factors
• Standard BS assumptions:
  • Flat nonrandom yield curve
  • The underlying asset has continuous IID-normal returns
• Our objective is to map risk in the underlying asset onto an option
  • This can be done using the option Delta and Gamma
VaR for bonds

- There are several ways to approximate bond risk as a function of risk in interest rates
- One way is to use Ito’s lemma, another to follow the derivation for options
- Here we only present the result, as a formal derivation would just repeat the one given for options
**Modified duration**

- We define *modified duration*, $D^*$, as the negative first derivative of the bond-pricing function, $g'(r)$, divided by prices:

$$D^* = -\frac{1}{P}g'(r)$$

- Modified duration measures price sensitivity of a bond to interest rate movements.
Duration-normal VaR

Two steps to calculate bond VaR

1. Identify the distribution of interest rate changes, $dr$
2. Map distribution onto bond prices
Duration-normal VaR

- We assume the distribution of interest rate changes is given by
  \[ r_t - r_{t-1} = dr \sim \mathcal{N} \left( 0, \sigma_r^2 \right) \]
  but we could use almost any distribution

- Regardless of whether we use Ito’s lemma or follow the derivation for options, we arrive at the duration-normal method to get bond VaR

- Here we find that bond returns are simply modified duration times interest rate changes so
  \[ R_{\text{Bond}} \overset{\text{Approximately}}{\sim} \mathcal{N} \left( 0, \left( \sigma_r D^* \right)^2 \right) \]
Duration-normal VaR

- Now the VaR follows directly:

\[
\text{VaR}_{\text{Bond}}(p) \approx D^* \times \sigma_r \times \Phi^{-1}(p) \times \vartheta
\]
Accuracy of duration-normal VaR

- The accuracy of these approximations depends on magnitude of duration and the VaR time horizon
- Main sources of error are assumptions of linearity and flat yield curve
- We now explore these issues graphically
Bond prices and duration

Accuracy of duration approximation for T=1

Interest rate

Bond price

Duration approximation
Bond prices and duration

Accuracy of duration approximation for T=50

Interest rate

Bond price
Duration approximation
Bond prices and duration
Accuracy of duration approximation for $T=1$ and $T=50$

- The graphs compare bond prices and duration approximation for two maturities, $T = 1$ and $T = 50$.
- It is clear that duration approximation is quite accurate for short-dated bonds, but very poor for long-dated ones.
- We conclude that maturity is a key factor when it comes to accuracy of VaR calculations using duration-normal methods.
Error in duration-normal VaR

Various volatilities of interest rate changes

\[
\text{Maturity} \quad \text{VaR (true)} \quad \text{VaR (duration)}
\]

\[
\sigma_r = 0.1\% \\
\sigma_r = 0.5\% \\
\sigma_r = 1.0\% \\
\sigma_r = 2.0\%
\]
Error in duration-normal VaR

Higher volatility of interest rate changes leads to larger error

- The graph on the previous slide shows how the accuracy of duration-normal VaR is affected by interest rate change volatility.
- Duration-normal VaR is compared with VaR (true), which is calculated with a Monte Carlo simulation.
- Looking at maturities from 1 year to 60 years and volatility from 0.1% to 2.0%, we see that the error in duration-normal VaR increases as volatility of interest rate changes increases.
Accurac of duration-normal VaR

- Based on these observations, we conclude that duration-normal VaR approximation is best for short-dated bonds and low volatilities.
- Quality declines sharply with increased volatility and longer maturities.
Convexity and VaR

• Straightforward to improve duration approximation by adding second-order term, thereby allowing for convexity
• However, even after incorporating convexity there is often considerable bias in VaR calculations
• Adding higher order terms increases mathematical complexity, especially if we have a portfolio of bonds
• For these reasons, Monte Carlo methods are generally preferred
Delta

• First-order sensitivity of an option with respect to the underlying price is called delta, defined as:

\[ \Delta = \frac{\partial g(P)}{\partial P} = \begin{cases} \Phi(d_1) > 0 & \text{call} \\ \Phi(d_1) - 1 < 0 & \text{put} \end{cases} \]

• Delta is equal to \( \pm 1 \) for deep-in-the-money options (depending on whether it is call or put), close to \( \pm 0.5 \) for at-the-money options and \( 0 \) for deep out-of-the-money options.
• A small change in $P$ changes the option price by approximately $\Delta$, but the approximation gets gradually worse as the deviation of $P$ becomes larger.

• We can graph the price of a call option for a range of strike prices and two different maturities to gauge the accuracy of the delta approximation.

• We let $X = 100$, $r = 0.01$ and $\sigma = 0.2$ and compare maturities of one and six months.
Accuracy of Delta approximation

One month

- Payoff at expiration
- Payoff one month to expiration
- Delta

Stock price

Option price

Payoff at expiration
Payoff one month to expiration
Delta
Accuracy of Delta approximation

**Six months**

- **Red line**: Payoff at expiration
- **Blue line**: Payoff six months to expiration
- **Orange line**: Delta

The graph shows the relationship between stock price and option price over a six-month period. The red line represents the payoff at expiration, the blue line represents the payoff six months to expiration, and the orange line represents the delta. The x-axis represents the stock price, ranging from $90 to $110, and the y-axis represents the option price, ranging from $0 to $100.
Gamma

• Second-order sensitivity of an option with respect to the underlying price is called gamma, defined as:

\[ \Gamma = \frac{\partial^2 g(P)}{\partial P^2} = e^{-r(T-t)} \frac{\Phi (d_1)}{P_t \sigma_a \sqrt{(T-t)}} \]

• Gamma is highest when an option is a little out of the money and dropping as the underlying price moves away from the strike price

• We can see this by adding a plot of gamma to the previous graph of option price with one month to expiry

  • Not surprising since the price plot increasingly becomes a straight line for deep in-the-money and out-of-the-money options
Gamma for the one month option

![Gamma graph for the one month option](image)

- Stock price
- Option price
- Gamma

<table>
<thead>
<tr>
<th>Stock price</th>
<th>Option price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$80</td>
<td>$0</td>
</tr>
<tr>
<td>$90</td>
<td>$5</td>
</tr>
<tr>
<td>$100</td>
<td>$10</td>
</tr>
<tr>
<td>$110</td>
<td>$15</td>
</tr>
<tr>
<td>$120</td>
<td>$20</td>
</tr>
</tbody>
</table>
Numerical example

- Consider an option that expires in six months \((T = 0.5)\) with strike price \(X = 90\), price \(P = 100\) and 20% volatility
- Let \(r = 5\%\) be the risk-free rate of return
- The call delta is 0.8395 and the put delta is \(-0.1605\)
- The gamma is 0.01724
Delta-normal VaR

- We can use delta to approximate changes in the option price as a function of changes in the price of the underlying.
- Denote daily change in stock prices as:

\[ dP = P_t - P_{t-1} \]

- The price change \( dP \) implies that the option price will change approximately by

\[ dg = g_t - g_{t-1} \approx \Delta dP = \Delta (P_t - P_{t-1}) \]

where \( \Delta \) is the option delta at time \( t-1 \); and \( g \) is either the price of a call or put.
• Simple returns on the underlying are

\[ R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \]

and following the BS assumptions, they are IID-normal with daily volatility \( \sigma_d \):

\[ R_t \sim \mathcal{N} \left( 0, \sigma_d^2 \right) \]

• The derivation of VaR for options parallels the one for simple returns in Chapter 5
Delta-normal VaR

Derivation of VaR for options

• Denote $\text{Var}_o(p)$ as the VaR of an option, where $p$ is probability:

$$p = \Pr (g_t - g_{t-1} \leq - \text{VaR}_o(p))$$

$$= \Pr (\Delta (P_t - P_{t-1}) \leq - \text{VaR}_o(p))$$

$$= \Pr (\Delta P_{t-1} R_t \leq - \text{VaR}_o(p))$$

$$= \Pr \left( \frac{R_t}{\sigma_d} \leq - \frac{1}{\Delta} \frac{\text{VaR}_o(p)}{P_{t-1} \sigma_d} \right)$$
### Delta-normal VaR

**Derivation of VaR for options**

- Now it follows that the VaR for holding an option on one unit of the asset is:

\[ \text{VaR}_o(p) \approx -|\Delta| \times \sigma_d \times \Phi^{-1}_R(p) \times P_{t-1} \]

- This means that the option VaR is simply \( \delta \) multiplied by the VaR of the underlying, \( \text{VaR}_u \):

\[ \text{VaR}_o(p) \approx |\Delta| \text{VaR}_u(p) \]

- We need absolute value because we may have put or call options and VaR is always positive
Quality of Delta-normal VaR

- The quality of this approximation depends on the extent of nonlinearities
  - Better for shorter VaR horizons
- For risk management purposes, poor approximation of delta to the true option price for large changes in the price of the underlying is clearly a cause of concern
**Delta and Gamma**

- We can also approximate the option price by the second-order expansion, $\Gamma$
- Since $dP$ is normal, $(dP)^2$ is chi-squared
- The same issues apply here as for bonds: Adding higher orders increases complexity a lot, without eliminating bias
Summary

- We have seen that forecasting VaR for options and bonds is much more complicated than for basic assets like stocks and foreign exchange.
- The mathematical complexity in this chapter is not high, but the approximations have low accuracy.
- To obtain higher accuracy the mathematics become much more complicated, especially for portfolios.
- This is why the Monte Carlo approaches in Chapter 7 are preferred in most practical applications.