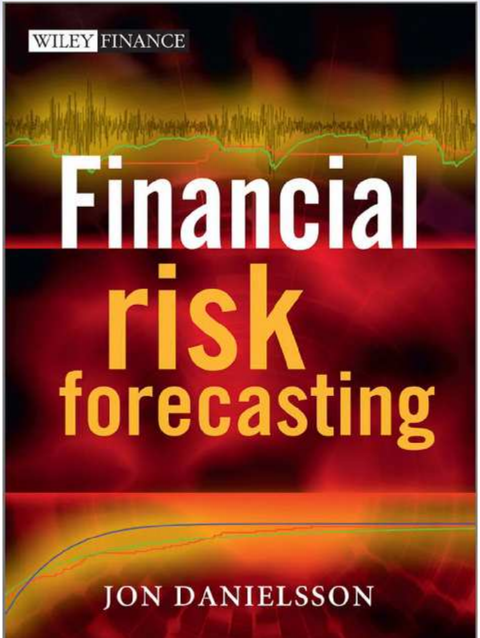
Financial Risk Forecasting
Chapter 6
Analytical Value-At-Risk For Options And Bonds

Jon Danielsson ©2023
London School of Economics

To accompany
Financial Risk Forecasting
www.financialriskforecasting.com
Published by Wiley 2011

Version 8.0, August 2023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Options

Options

- An option gives its owner the right, but not the obligation, to *call* (buy) or *put* (sell) an underlying asset at a *strike price* on a fixed expiry date
    - European options can only be exercised at expiration
    - American options can be exercised at any point up to expiration
- We will focus on European options, but the basic analysis could be extended to many other variants

Black-Scholes Equation for European Options

- Black and Scholes (1973) developed an equation for pricing European options
- Refer to the Black-Scholes (BS) pricing function as $g(\cdot)$
- We use the following notation:

P_t	Price of underlying asset
X	Strike price
r	Annual risk-free interest rate
$T-t$	Time until expiration
σ_a	Annual volatility
Φ	Standard normal distribution

Black-Scholes Mathematics

- The BS function for an European option

$$\text{put}_t = Xe^{-r(T-t)} - P_t + \text{call}_t$$

$$\text{call}_t = P_t \Phi\left(d_1\right) - Xe^{-r(T-t)} \Phi\left(d_2\right)$$

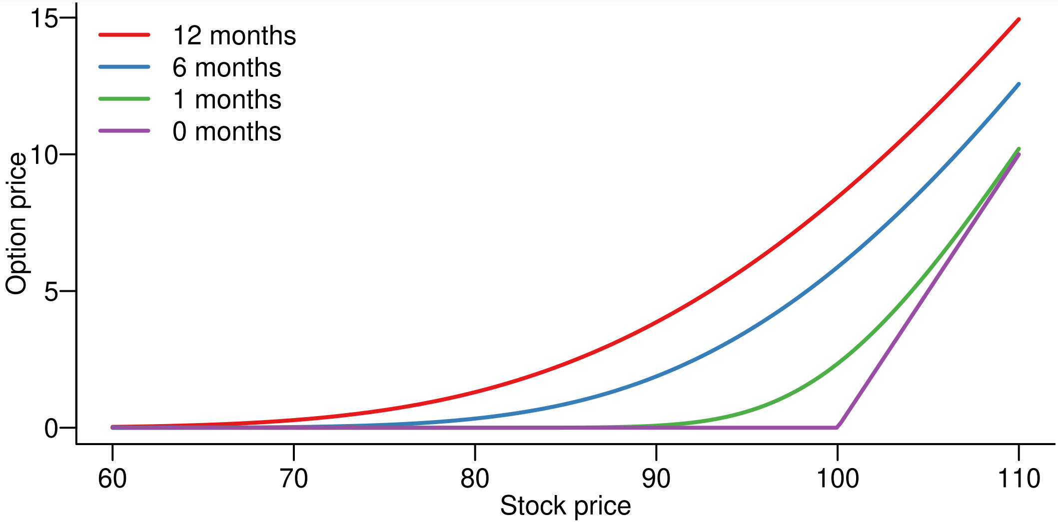
where

$$d_1 = \frac{\log\left(P_t/X\right) + \left(r + \sigma_a^2/2\right)(T-t)}{\sigma_a \sqrt{T-t}}$$

$$d_2 = \frac{\log\left(P_t/X\right) + \left(r - \sigma_a^2/2\right)(T-t)}{\sigma_a \sqrt{T-t}} = d_1 - \sigma_a \sqrt{T-t}$$

Black-Scholes Equation

- Value of an option is affected by many underlying factors
- Standard BS assumptions
  - Flat non-random yield curve
  - The underlying asset has continuous IID normal returns
- Our objective is to map risk in the underlying asset onto an option
  - This can be done using the option *Delta* and *Gamma*


12 months
6 months
1 months
0 months
Option price
0
5
10
15
60
70
80
90
100
110
Stock price

VaR For Options

Delta

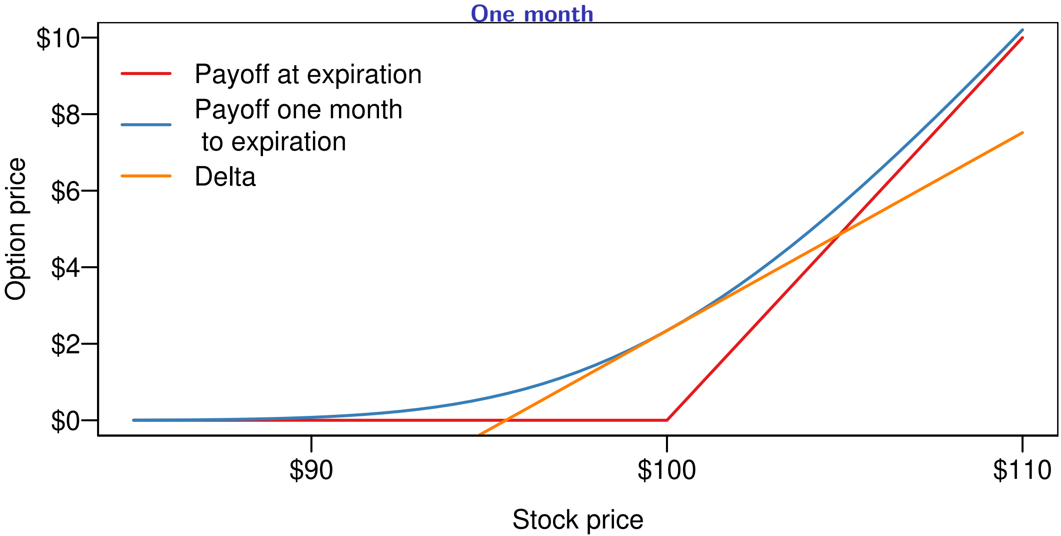
- First-order sensitivity of an option with respect to the underlying price is called delta, defined as:

$$\Delta = \frac{\partial g(P)}{\partial P} = \begin{cases} \Phi\left(d_1\right) > 0 & \text{call} \\ \Phi\left(d_1\right) - 1 < 0 & \text{put} \end{cases}$$

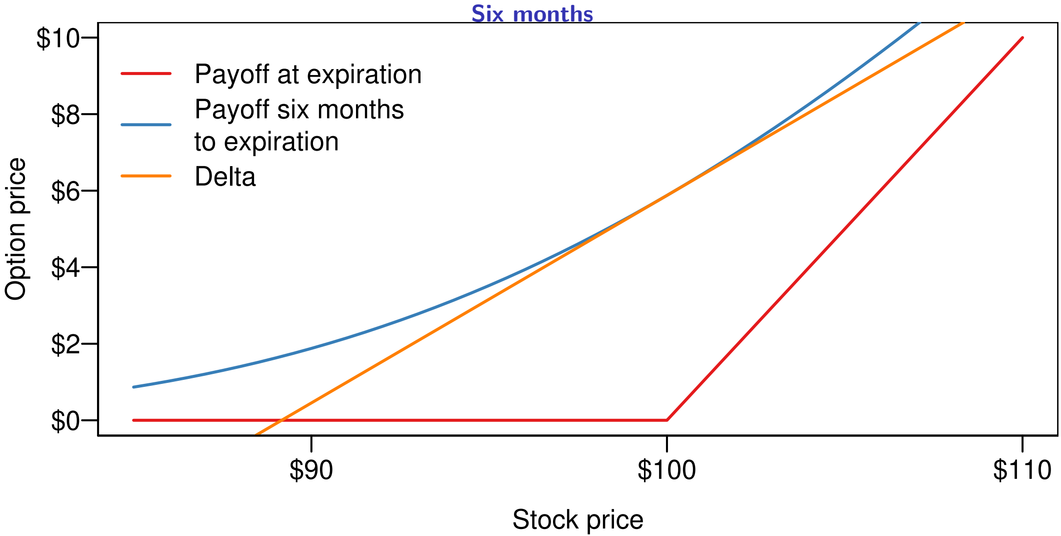
- Delta is equal to $+/-$ 1 for deep-in-the-money options (depending on whether it is call or put), close to $+/-$ 0.5 for at-the-money options and 0 for deep out-of-the-money options

- A small change in price changes the option price by approximately Δ, but the approximation gets gradually worse as the price deviation becomes larger
- We can graph the price of a call option for a range of strike prices and two different maturities to gauge the accuracy of the delta approximation
- We let $X = 100$, $r = 0.01$ and $\sigma_a = 0.2$ and compare maturities of one and six months

Accuracy of Delta Approximation



Accuracy of Delta Approximation



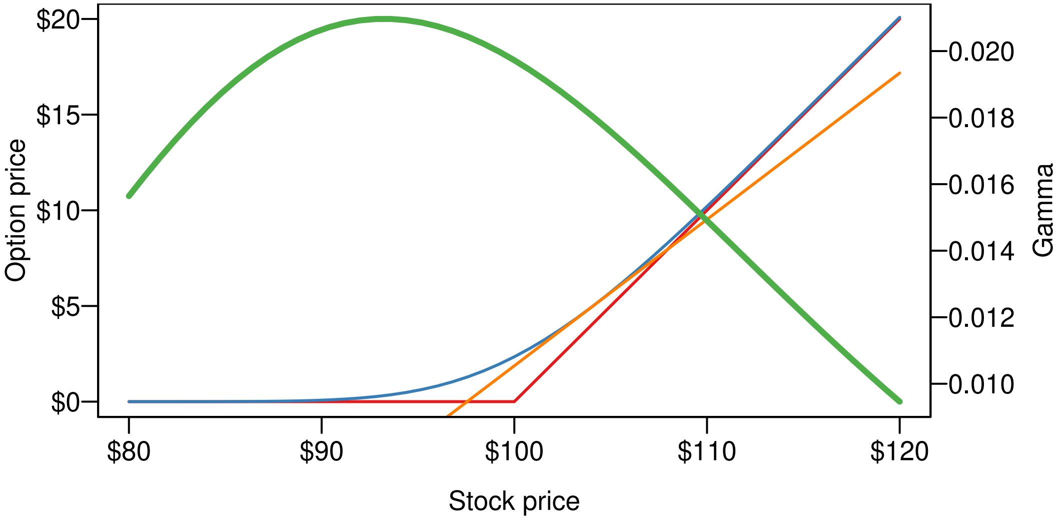
Gamma

- Second-order sensitivity of an option with respect to the underlying price is called gamma, defined as:

$$\Gamma = \frac{\partial^2 g(P)}{\partial P^2}$$

- Gamma is highest when an option is a little out of the money and dropping as the underlying price moves away from the strike price
- We can see this by adding a plot of gamma to the previous graph of option price with one month to expiry
  - Not surprising since the price plot increasingly becomes a straight line for deep in-the-money and out-of-the-money options

Gamma for the One Month Option



Numerical Example

- Consider an option that expires in six months ($T = 0.5$) with strike price $X = 90$, price $P = 100$ and 20% volatility
- Let $r = 5\%$ be the risk-free rate of return
- The call delta is 0.8395 and the put delta is -0.1605
- The put gamma is 0.01724

Delta-Normal VaR

- We can use delta to approximate changes in the option price as a function of changes in the price of the underlying
- Denote daily change in stock prices as:

$$dP = P_t - P_{t-1}$$

- The price change dP implies that the option price will change approximately by

$$dg = g_t(\cdot) - g_{t-1}(\cdot) \approx \Delta dP = \Delta\left(P_t - P_{t-1}\right)$$

where Δ is the option delta at time $t-1$; and g is either the price of a call or put

- Simple returns on the underlying are

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

and following the BS assumptions, they are IID-normal with daily volatility σ_d:

$$R_t \sim \mathcal{N}\left(0, \sigma_d^2\right)$$

- The derivation of VaR for options parallels the one for simple returns in Chapter 5

Delta-Normal VaR

Derivation of VaR for options

- Denote $\text{Var}_o(\rho)$ as the VaR of an option

$$
\begin{aligned}
p =& \mathbb{P}\left(g_t - g_{t-1} \leq -\text{VaR}_o(\rho)\right) \\
=& \mathbb{P}\left(\Delta\left(P_t - P_{t-1}\right) \leq -\text{VaR}_o(\rho)\right) \\
=& \mathbb{P}\left(\Delta P_{t-1} R_t \leq -\text{VaR}_o(\rho)\right) \\
=& \mathbb{P}\left(\frac{R_t}{\sigma_d} \leq -\frac{1}{\Delta}\frac{\text{VaR}_o(\rho)}{P_{t-1}\sigma_d}\right)
\end{aligned}
$$

Delta-Normal VaR

Derivation of VaR for options

- Now it follows that the VaR for holding an option on one unit of the asset is:

$$\mathrm{VaR}_o(\rho) \approx -|\Delta| \times \sigma_d \times \Phi_R^{-1}(\rho) \times P_{t-1}$$

- This means that the option VaR is simply δ multiplied by the VaR of the underlying, VaR_u:

$$\mathrm{VaR}_o(\rho) \approx |\Delta| \, \mathrm{VaR}_u(\rho)$$

- We need absolute value because we may have put or call options and VaR is always positive

Quality of Delta-Normal VaR

- The quality of this approximation depends on the extent of non-linearities
  - Better for shorter VaR horizons
- For risk management purposes, poor approximation of delta to the true option price for large changes in the price of the underlying is clearly a cause of concern

Delta and Gamma

- We can also approximate the option price by the second-order expansion, Γ
- Since dP is normal, $(dP)^2$ is chi-squared
- The same issues apply here as for bonds: Adding higher orders increases complexity a lot, without eliminating bias

Summary

- We have seen that forecasting VaR for options and bonds is much more complicated than for basic assets like stocks and foreign exchange
- The mathematical complexity in this chapter is not high, but the approximations have low accuracy
- To obtain higher accuracy the mathematics become much more complicated, especially for portfolios
- This is why the Monte Carlo approaches in Chapter 7 are preferred in most practical applications