Financial Risk Forecasting
Chapter 7
Simulation methods for VaR for options and bonds

Jon Danielsson ©2020
London School of Economics

To accompany
Financial Risk Forecasting
www.financialriskforecasting.com
Published by Wiley 2011
Version 5.0, August 2020
The focus of this chapter

- Chapter 6 demonstrates limitations of analytical VaR methods for options and bonds
- The focus in this chapter is on simulation methods, sometimes called *Monte Carlo (MC) simulation*
  1. Pseudo random number generators (RNG)
  2. Simulation pricing of options and bonds
  3. Simulation VaR for one asset and portfolios
  4. Issues in Monte Carlo estimation
Idea

- Replicate a part of the world in computer software
- For example market outcomes, based on some model of market evolution
- Sufficient number of simulations (replications) ideally yield a large and representative sample of market outcomes
- Use that to calculate quantities of interest (e.g. VaR)
Calendar time and trading time

- We use two different measures of time
  
  **Calendar time** (365/6 days) used for interest rate calculations
  
  **Trading time** (∼ 250 days) used for risk calculations

- This is because we earn interest every day
- But calculate volatilities only from days (trading days) when stock exchanges open (Mondays to Fridays, excluding holidays)
- R will allow precise date calculations and has a database of dates when various exchanges are open
Notation

\begin{align*}
F & \quad \text{Futures price} \\
g & \quad \text{Derivative price} \\
S & \quad \text{Number of simulations} \\
x^b & \quad \text{Portfolio holdings (basic assets)} \\
x^o & \quad \text{Portfolio holdings (derivatives)}
\end{align*}
Simulation pricing of bonds
Bond pricing

• Price and risk of fixed income assets (e.g. bonds) is based on market interest rates
• Using a model of the distribution of interest rates, we can simulate random yield curves and obtain the distribution of bond prices
• We map distribution of interest rates to the distribution of bond prices
Analytical bond pricing

- Denote $r_t$ the annual interest rate at time $t$
- The present value of a bond is the present discounted value of its cash flows:

$$P = \sum_{t=1}^{T} \frac{\tau_t}{(1 + r_t)^t}$$

- Where $P$ is bond price and $\tau_t$ is cash flow at time $t$
Analytical bond pricing

- Suppose we have a bond with 10 years to expiration, 7% annual interest, $10 par value, and current market rates:

\[
\{r_t\}_{t=1}^{10} = (5.00, 5.69, 6.09, 6.38, 6.61, 6.79, 7.07, 7.19, 7.30) \times 0.01
\]

- The bond has a current value of $9.91
Simulated bond pricing

- Assume here that the yield curve can only shift up and down, not change shape
- Shocks to yields, $\epsilon_i$

$$\epsilon_i \sim \mathcal{N} (0, \sigma^2)$$

- For the sake of demonstration we set the number of simulations as $S = 8$, but note that accurate estimates require much more simulations
Eight yield curve simulations
Eight yield curve simulations

![Graph showing eight yield curve simulations](image-url)
Eight yield curve simulations
Eight yield curve simulations

![Graph showing eight yield curve simulations](image)

- **Yield** vs **Time**
  - True line
  - Simulated lines

- **Time**轴从2到10
- **Yield**轴从4%到9%

---

Eight yield curve simulations
Simulated bond pricing

- The equation for the $i^{th}$ simulated price, $P_i$, now becomes

$$P_i = \sum_{t=1}^{T} \frac{T_t}{(1 + r_t + \epsilon_i)^t}$$

where $r_t + \epsilon_i$ is the $i^{th}$ simulated interest rate at time $t$

- Compare the eight bond prices obtained with $S = 8$ yield curve simulations with the distribution of bond prices when $S = 50,000$
Eight bond price simulations

Simulation

Simulated price

True price

$0

$2

$4

$6

$8

$10

1 2 3 4 5 6 7 8

Simulation
Density of simulated bond prices

Normal distribution superimposed, $S = 50,000$
Allowing yield curve to change shape

• Key assumptions:
  • Yield curve only shifts up and down
  • Distribution of interest rate changes is normal

• The assumptions may be unrealistic in practice but it is relatively straightforward to relax them
  • in practice it rotates and twists
  • can use principal components (PCA)
Simulation pricing of options
Simulation approach

- The price of an asset be the expectation of its final payoff under *risk neutrality*
- Depends on the price movements of its underlying asset
- If sufficient number of price paths are simulated
- Obtain an estimate of the true price
Option pricing

- Get price of European options on non-dividend-paying stock where all Black-Scholes (BS) assumptions hold
- Two primitive assets in BS pricing model:
  - *risk-free asset* with instantaneous rate $r$
  - Underlying stock, follows normally distributed random walk with drift $r$ (geometric Brownian motion in continuous time)
- The no-arbitrage futures price of stock for delivery at time $T$ is given by:
  \[ F = Pe^{rT} \]
Analytical option pricing

• Suppose we have a European call option with
  1. current stock price $50
  2. 20% annual volatility
  3. 5% annual risk-free rate
  4. 6 months to expiration
  5. $40 strike price

• The price is $11.0873
Simulated option pricing

- We simulate returns until expiration and use these values to calculate simulated futures prices.
- With sufficient sample of futures prices we can compute the set of payoffs of the option.
- The MC price is then given by the mean of these payoffs.
Simulated option pricing

- The only complexity is due to expectation of a log-normal RN, i.e. if
  
  $O \sim \mathcal{N}(\mu, \sigma^2)$

  then:

  $E[\exp(O)] = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$

- We apply a *log-normal correction* (subtract $\frac{1}{2}\sigma^2$ from simulated stock return) to ensure that expectation of simulated returns is the same as theoretical value

- See density of $S = 10^6$ futures prices and option payoffs using same values as in the example before
Density of simulated futures prices

\[ S = 10^6, \text{ normal distribution superimposed} \]
Density of simulated option prices

Based on simulated futures prices with $S = 10^6$

Black–Scholes price
Simulated option pricing

Numerical example

- Mean of simulated option prices is the MC price
  - In this case R gives $11.08709, close enough to the Black-Scholes price of $11.0873
- Note the asymmetry in the density of simulated futures prices (result of log-normal distribution of prices)
- The VaR can be read off the previous graph, e.g. 1% smallest value of distribution gives 99% VaR
Simulation of VaR
Simulating a stock price using continuous time

- The simplest model of a stock price in continuous time is geometric Brownian motion

\[ dS = \mu S \, dt + \sigma S \, dz \]

- Then, returns are

\[ y_t = \log p_t - \log p_{t-1} \]

- If this is not what we will do here, instead, use arithmetic returns

\[ R_t = \frac{P_t}{P_t} - 1 \]

- i.e.

\[ P_t = R_{t-1}(R_t + 1) \]
Simulation of VaR for one asset

- Simulate one-day return of an asset
- Apply analytical pricing formulas to simulated future price
- Obtain simulated profits/losses ($\text{P/L}$) as difference between tomorrow’s simulated future values and today’s known value
- Calculate MC VaR from simulated $\text{P/L}$
Setup

- Consider asset with price $P_t$ and IID normal returns, with one-day volatility $\sigma$ and risk-free rate $r$ (in continuous time)
- Number of units of basic asset held in a portfolio is denoted by $x^b$, while $x^o$ indicates number of options held
- Note that it is the $t + 1$ price that will be simulated
We will go through a series of ever more complicated examples

1. Simulation of VaR for one asset (no option)
2. Simulation of VaR for one option
3. Simulation of VaR for a portfolio of one option and one stock
1. VaR with one basic asset

Six-step procedure for obtaining MC VaR

1. Compute initial portfolio value:

\[ \vartheta_t = x^b P_t \]
1. VaR with one basic asset

Six-step procedure for obtaining MC VaR

1. Compute initial portfolio value:

\[ \vartheta_t = x^b P_t \]

2. Simulate \( S \) one-day returns

\[ R_{t+1,i} \sim \mathcal{N}(0, \sigma^2), \quad i = 1, \ldots, S \]
1. VaR with one basic asset

Six-step procedure for obtaining MC VaR

3. Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+1,i}) \]

4. Calculate the simulated futures value of the portfolio:

\[ \vartheta_{t+1,i} = x^b P_{t+1,i} \]
1. VaR with one basic asset

Six-step procedure for obtaining MC VaR

3. Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+1,i}) \]

4. Calculate the simulated futures value of the portfolio:

\[ \vartheta_{t+1,i} = x^b P_{t+1,i} \]
1. VaR with one basic asset

Six-step procedure for obtaining MC VaR

3 Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+1,i}) \]

4 Calculate the simulated futures value of the portfolio:

\[ \vartheta_{t+1,i} = x^b P_{t+1,i} \]

5 The \( i^{th} \) simulated P/L is then:

\[ q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t \]
1. VaR with one basic asset

Six-step procedure for obtaining MC VaR

3. Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+1,i}) \]

4. Calculate the simulated futures value of the portfolio:

\[ \vartheta_{t+1,i} = x^b P_{t+1,i} \]

5. The \( i \)th simulated P/L is then:

\[ q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t \]

6. VaR can be obtained directly from the vector of simulated P/L, \( \{ q_{t+1,i} \}_i \), e.g. VaR(0.01) is the 1% smallest value
2. VaR with an option

Modified six-step procedure

- For options we need to modify the procedure
- Let $g(\cdot)$ denote the Black-Scholes equation and suppose we have $x^o$ options
- We replace steps 1 and 4 and come up with the following procedure
2. **VaR with an option**

Modified six-step procedure

1’ Initial portfolio is

\[ \varphi_t = x^0 g \left( P_t, X, T, \sqrt{250} \sigma, r \right) \]
2. VaR with an option

Modified six-step procedure

1’ Initial portfolio is

\[ \varphi_t = x^0 g \left( P_t, X, T, \sqrt{250} \sigma, r \right) \]

2 Simulate \( S \) one-day returns

\[ R_{t+1,i} \sim \mathcal{N} \left( 0, \sigma^2 \right), \quad i = 1, \ldots, S \]
2. **VaR with an option**

*Modified six-step procedure*

3. Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+,i}) \]
2. VaR with an option

Modified six-step procedure

3 Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+1,i}) \]

4’ The \( i \)th simulated future value of the portfolio is

\[ \vartheta_{t+1,i} = \chi^0 g \left( P_{t+1,i}, X, T - \frac{1}{365}, \sqrt{250\sigma}, r \right) \]
2. VaR with an option

Modified six-step procedure

5 The $i^{th}$ simulated P/L is then:

$$q_{t+1,i} = \theta_{t+1,i} - \theta_t$$
2. VaR with an option

Modified six-step procedure

5 The \( i \)th simulated P/L is then:

\[
q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t
\]

6 VaR can be obtained directly from vector of simulated P/L, \( \{q_{t+1,i}\}_{i=1}^S \), e.g. VaR(0.01) is 1% smallest value
### 2. MC VaR of option

- One call option with strike price $X = 100$ and 3 months to expiry
- R and Matlab both give VaR(0.01) of $1.21$
2. Density of simulated P/L

Normal distribution superimposed

Profit/Loss

Density

VaR 95%
3. VaR with an options and a stock

Modified six-step procedure

- Now consider the case of a portfolio with both a stock and option(s) on the same stock
- Suppose we only have one type of option
- As in the case where we only had one option on a basic asset, we replace steps 1 and 4
3. VaR with an options and a stock

Modified six-step procedure

1” Initial portfolio is

\[ \varphi_t = x^b P_t + x^o g \left( P_t, X, T, \sqrt{250} \sigma, r \right) \]
3. VaR with an options and a stock

Modified six-step procedure

1” Initial portfolio is

$$v_t = x^b P_t + x^o g \left( P_t, X, T, \sqrt{250}\sigma, r \right)$$

2 Simulate $S$ one-day returns

$$R_{t+1,i} \sim \mathcal{N} \left( 0, \sigma^2 \right), \quad i = 1, \ldots, S$$
3. VaR with an options and a stock

Modified six-step procedure

3 Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+,i}) \]
3. VaR with an options and a stock

Modified six-step procedure

3 Calculate one-day future price:

\[ P_{t+1,i} = P_t \times (1 + R_{t+,i}) \]

4” The \( i \)th simulated future value of the portfolio is

\[ \vartheta_{t+1,i} = x^b P_{t+1,i} + x^o g \left( P_{t+1,i}, X, T - \frac{1}{365}, \sqrt{250} \sigma, r \right) \]
3. VaR with an options and a stock

Modified six-step procedure

5 The \(i^{th}\) simulated P/L is then:

\[
q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t
\]
3. VaR with an options and a stock

Modified six-step procedure

5 The \( i^{th} \) simulated P/L is then:

\[
q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t
\]

6 VaR can be obtained directly from vector of simulated P/L, \( \{q_{t+1,i}\}_i^S \), e.g. VaR(0.01) is 1% smallest value
<table>
<thead>
<tr>
<th>Bonds</th>
<th>Options</th>
<th>One asset VaR</th>
<th>Portfolio VaR</th>
<th>Simulation issues</th>
</tr>
</thead>
</table>

Simulation pricing of a portfolio
Simulation of portfolio VaR

• Consider the multivariate case, i.e. the case of more than one underlying assets
  • Main difference: We need to simulate correlated returns for all assets
  • Simulated future prices calculated as before and portfolio value obtained by summing up individual simulated asset holdings
Simulation of portfolio VaR

- Suppose we have two non-derivative assets with daily return distribution

\[ \mathcal{N}\left(\mu = \begin{pmatrix} 0.05/365 \\ 0.05/365 \end{pmatrix}, \Sigma = \begin{pmatrix} 0.01 & 0.0005 \\ 0.0005 & 0.02 \end{pmatrix}\right) \]

- Let \( x^b \) be a vector of holdings
Notation

- The notation becomes cluttered for the multivariate case
- Now we have to denote variables by time period, asset and simulation
- We let $P_{t,k,i}$ denote the $i^{th}$ simulated price of asset $k$ at time $t$, that is:

$$P_{\text{time,asset,simulation}} = P_{t,k,i}$$
Portfolio VaR for basic assets

Six-step procedure for obtaining MC portfolio VaR

1. Compute initial portfolio value:

\[ \vartheta_t = \sum_{k=1}^{K} \chi_k^b P_{t,k} \]
Portfolio VaR for basic assets

Six-step procedure for obtaining MC portfolio VaR

1. Compute initial portfolio value:

\[ \vartheta_t = \sum_{k=1}^{K} x_k^b P_{t,k} \]

2. Simulate a vector of one-day returns from today to tomorrow

\[ R_{t+1,i} \sim \mathcal{N}\left( \mu - \frac{1}{2} \text{Diag}\Sigma, \Sigma \right) \]

Diag\Sigma extracts the diagonal elements of \( \Sigma \) (because of log-normal correction)
**Portfolio VaR for basic assets**

**Six-step procedure for obtaining MC portfolio VaR**

3. The $i^{th}$ simulated future price of asset $k$ is:

\[ P_{t+1,k,i} = P_{t,k} (1 + R_{t+1,k,i}) \]
Portfolio VaR for basic assets

Six-step procedure for obtaining MC portfolio VaR

3 The $i^{th}$ simulated future price of asset $k$ is:

$$P_{t+1,k,i} = P_{t,k} (1 + R_{t+1,k,i})$$

4 The $i^{th}$ simulated futures value of the portfolio is:

$$\vartheta_{t+1,i} = \sum_{k=1}^{K} \chi_k^b P_{t+1,k,i}$$
Portfolio VaR for basic assets

Six-step procedure for obtaining MC portfolio VaR

5 The $i^{th}$ simulated P/L is then:

$$q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_{t}$$
Portfolio VaR for basic assets

Six-step procedure for obtaining MC portfolio VaR

5 The \( i \)th simulated P/L is then:

\[
q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t
\]

6 VaR can be obtained directly from vector of simulated P/L, \( \{q_{t+1,i}\}_{i=1}^S \), as before
Portfolio VaR for options

Modified six-step procedure

- For options we need to modify steps 1 and 4 from the procedure outlined above
  - Similar to modifications for the univariate case before
- For simplicity suppose the portfolio has only one type of option type per stock
1' Initial portfolio is

\[ \vartheta_t = \sum_{k=1}^{K} \left( x_k^b P_{t,k} + x_k^o g \left( P_{t,k}, X_k, T, \sqrt{250} \sigma_k, r \right) \right) \]
Portfolio VaR for options

Modified six-step procedure

1' Initial portfolio is

\[ \vartheta_t = \sum_{k=1}^{K} \left( x_k^b P_{t,k} + x_k^o g \left( P_{t,k}, X_k, T, \sqrt{250} \sigma_k, r \right) \right) \]

2 Simulate a vector of one-day returns from today to tomorrow

\[ R_{t+1,i} \sim \mathcal{N} \left( \mu - \frac{1}{2} \text{Diag} \Sigma, \Sigma \right) \]

Diag \Sigma extracts the diagonal elements of \( \Sigma \) (because of the log-normal correction)
Portfolio VaR for options

Modified six-step procedure

3 The $i^{th}$ simulated future price of asset $k$ is:

$$P_{t+1,k,i} = P_{t,k} (1 + R_{t+1,k,i})$$
Portfolio VaR for options

Modified six-step procedure

3. The $i^{th}$ simulated future price of asset $k$ is:

$$P_{t+1,k,i} = P_{t,k} (1 + R_{t+1,k,i})$$

4’. The $i^{th}$ simulated future value of the portfolio is

$$\vartheta_{t+1,i} = \sum_{k=1}^{K} (x^b P_{t+1,i} + x^o g \left( P_{t+1,k,i}, X_k, T - \frac{1}{365}, \sqrt{250}\sigma_k, r \right))$$
Portfolio VaR for options
Modified six-step procedure

5 The $i^{th}$ simulated P/L is then:

$$q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t$$
5 The $i^{th}$ simulated P/L is then:

$$q_{t+1,i} = \vartheta_{t+1,i} - \vartheta_t$$

6 VaR can be obtained directly from vector of simulated P/L, $\{q_{t+1,i}\}^S_{i=1}$, as before
Richer versions

- We used simple examples to avoid cluttered notation, straightforward to allow for more complicated portfolios
  - Number of stocks and multiple options on each stock
  - American (or more exotic) options
  - Combination of fixed income assets with stocks and options
- Also, we could use other distributions (e.g. Student-\(t\) or even historical simulation)
Issues in simulation estimation
Simulation issues

• Several issues need to be addressed in all MC exercises, of which two are most important:
  1. Quality of RNG and transformation method
  2. Number of simulations
Quality of RNG

- MC simulation is not only dependent on quality of the underlying stochastic model, also depends on quality of the RNG used
- Low-quality generators give biased or inaccurate results
  - E.g. a simulation size of 100 with period of 10 will repeat same calculation 10 times
- Complicated portfolios may demand large number of RNs and therefore high-quality RNGs
Quality of RNG

- Many transformation methods are only optimally tuned for the center of the distribution
- This becomes particularly problematic when simulating extreme events
- Some transformation methods use linear approximations for extreme tails, which leads to extreme uniforms being incorrectly transformed
Choosing number of simulations

- Choosing appropriate number of simulations is important
  - Too few give inaccurate answers
  - Too many waste time and computer resources
- In special cases formal statistical tests provide guidance, but usually informal methods have to be relied upon
- It is sometimes stated that accuracy of simulations is related to inverse simulation size
  - This is based on assumption of linearity, which is not correct for the problems in this chapter
Choosing number of simulations

- Best way is to simply increase number of simulations and see how MC estimate converges
- Rule of thumb: Sufficient simulation size when numbers have stopped changing up to three significant digits
- We can also compare *convergence of MC estimate* to the true (analytical) price
Convergence of MC estimate
Comparison with analytical Black-Scholes price

• In a example on slide 19 we computed analytical call price of $11.0873 for a European option
• Now calculate MC estimates for different simulation sizes and compare the results with the true (analytical) price
Cumulative MC estimates
Comparison with analytical Black-Scholes price

Black–Scholes price

Simulations
Cumulative MC estimates

Comparison with analytical Black-Scholes price

![Graph showing cumulative MC estimates compared to analytical Black-Scholes price](image)
Cumulative MC estimates

Comparison with analytical Black-Scholes price

Average

11.1
11.0
10.9
10.8
10.7

Simulations

$10^2$ $10^3$ $10^4$ $10^5$ $10^6$
Cumulative MC estimates

Comparison with analytical Black-Scholes price
Cumulative MC estimates
Comparison with analytical Black-Scholes price
Convergence of MC estimate
Comparison with analytical Black-Scholes price

- Based on graph on previous slide, it seems to take about 5000 simulations to get three significant digits correct
- However, there are still fluctuations in the estimate for 5 million simulations
Convergence of MC VaR estimate

- Look at the convergence of MC VaR estimates as the simulation size increases
- Graph MC VaR for a stock with daily volatility 1% along with ±99% confidence intervals
Convergence of MC VaR estimates

With ±99% confidence intervals

Simulations

VaR

115
110
105
100
95
90
85

10^2  50^2  50^3  50^4
Convergence of MC VaR estimates

With ±99% confidence intervals
Convergence of MC VaR estimates

With ±99% confidence intervals

Simulations
Convergence of MC VaR estimates

With $\pm 99\%$ confidence intervals